Monday Night Calculus

Strategy for Testing Series

Exercises

1. Determine whether each series converges or diverges. Explain you reasoning.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$

Use the Ratio Test with $a_n = \dfrac{2^n}{n!}$.

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{2^{n+1}}{(n+1)!}}{\dfrac{2^n}{n!}} = \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \frac{2}{n+1}$$

$$\lim_{n\to\infty} \frac{2}{n+1} = 0 < 1$$

Therefore by the Ratio Test, the series is absolutely convergent, and therefore convergent.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{2+n}{2n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2+n}{2n} = \lim_{n\to\infty} \frac{\frac{2}{n}+1}{2} = \frac{1}{2} \neq 0$$

The series diverges by the Test for Divergence.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2}$

The function $f(x) = \dfrac{2}{x^2}$ is continuous, positive, and decreasing on $[1, \infty)$.

Use the Integral Test.

$$\int_1^{\infty} \frac{2}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{2}{x^2}\,dx = \lim_{b\to\infty} \left[-\frac{2}{x}\right]_1^b$$

$$= \lim_{b\to\infty} \left[-\frac{2}{b} + \frac{2}{1}\right] = 2$$

Since the integral is convergent, the series is convergent, by the Integral Test.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

This is an alternating series. Let $b_n = \dfrac{1}{\sqrt{n}}$.

$$b_{n+1} = \frac{1}{\sqrt{n+1}} \leq \frac{1}{\sqrt{n}} = b_n \text{ for all } n$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

The series is convergent by the Alternating Series Test.

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{\sqrt{n}}$

$$\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

The series diverges: a p-series with $p = \dfrac{1}{2} \leq 1$

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

Use the Ratio test with $a_n = \dfrac{n^2}{2^n}$

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{(n+1)^2}{2^{n+1}}}{\dfrac{n^2}{2^n}} = \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} = \frac{1}{2}\left(\frac{n+1}{n}\right)^2$$

$$\lim_{n\to\infty} \frac{1}{2}\left(\frac{n+1}{n}\right)^2 = \frac{1}{2} < 1$$

Therefore by the Ratio Test, the series is absolutely convergent, and therefore convergent.

2. For $n = 1, 2, 3, \ldots$, let $a_{2n} = \dfrac{1}{n}$ and $a_{2n-1} = -\dfrac{1}{2^n}$, so that

$$\sum_{i=1}^{\infty} a_i = -\frac{1}{2} + 1 - \frac{1}{4} + \frac{1}{2} - \frac{1}{8} + \frac{1}{3} - \frac{1}{16} + \frac{1}{4} + \cdots$$

Does the alternating series test apply? Does this series converge or diverge?

The alternating series test does not apply. The terms do alternate in sign and the terms approach 0, but the terms do not decrease in absolute value monotonically.

This series is actually a perfect *shuffling* of the negative of a convergent geometric series ($r = 1/2$) and the harmonic series. Since the harmonic series diverges, the given series also diverges.

3. Let f be the function defined by $f(x) = 2x^3 - 5x^2 + 7x - 6$.

(a) Find the second degree Taylor polynomial, q_0, or f about $x = 0$ and the second degree Taylor polynomial, q_2 for f about $x = 2$. Graph f, q_0, and q_2 on the same coordinate axes.

$f(x) = 2x^3 - 5x^2 + 7x - 6 : f(0) = -6, \;\; f(2) = 4$

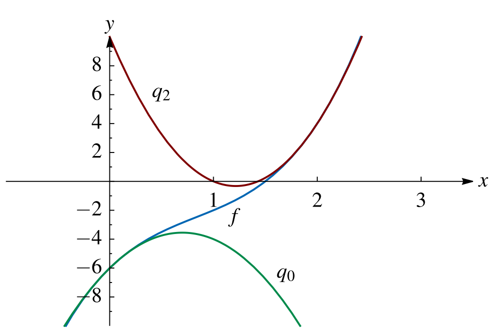
$f'(x) = 6x^2 - 10x + 7 : \;\; f'(0) = 7, \;\; f'(2) = 11$

$f''(x) = 12x - 10 : \;\; f''(0) = -10, \;\; f''(2) = 14$

$f'''(x) = 12, \;\; f^{(n)} = 0$, for $n \geq 4$

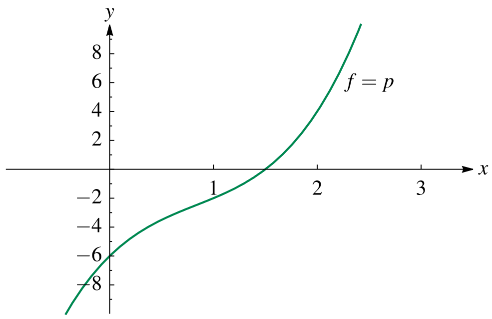
$$q_0(x) = -6 + 7x - \frac{10}{2!}x^2 = -6 + 7x - 5x^2$$

$$q_2(x) = 4 + 11(x-2) + 7(x-2)^2$$



(b) Find the third degree Taylor polynomial, p, for f about $x = 2$. Graph p and f on the same coordinate axes.

$$p(x) = 4 + 11(x-2) + 7(x-2)^2 + 2(x-2)^3$$



(c) Use the LaGrange error bound to explain why an nth degree Taylor polynomial approximation about any value $x = a$ to an nth degree polynomial function will always be an exact match.

$$|R_n(x)| \le \max_{z \text{ between } x \text{ and } a} |f^{(n+1)}(z)| \cdot \frac{|x-a|^{n+1}}{(n+1)!}$$

If f is an nth degree polynomial, then $f^{(n+1)}(x) = 0$ and the LaGrange error bound is 0. The nth degree Taylor polynomial is an exact match.

4. The Maclaurin series for $\sin x$ is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

(a) Use the first 5 nonzero terms of the series to approximate $\sin(1.0)$.

$$\sin(1) \approx 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \frac{1}{9!} = 0.841471$$

(b) What is the alternating series error bound for the approximation found in part (a)?

$$\text{Error} \le \frac{1}{11!} = 0.0000000250521$$

(c) What is the Lagrange error bound for the approximation found in part (a)?

The ninth degree MacLaurin polynomial for $\sin x$ is the same as the tenth degree MacLaurin polynomial since the coefficient of the tenth degree term is 0, so $n = 10$.

All derivatives of $\sin x$ are bounded by -1 and 1.

$$|R_n(x)| \le 1 \cdot \frac{|1-0|^{11}}{11!} = 0.0000000250521$$

Bonus Check:

$\sin 1 = 0.841470984808$ (to 12 decimal places)

$1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \frac{1}{9!} = 0.841471009700$

$|0.841470984808 - 0.8414710097| = 0.000000024892$ WOW!

5. Use the Integral Test error bound to estimate the error in using $\sum_{n=1}^{10} \frac{1}{n^3}$ to approximate $\sum_{n=1}^{\infty} \frac{1}{n^3}$

$$s_{10} = \sum_{n=1}^{10} \frac{1}{n^3} = 1.19753$$

$$\int_{11}^{\infty} \frac{1}{x^3}\, dx = \frac{1}{242} = 0.004132$$

$$\int_{10}^{\infty} \frac{1}{x^3}\, dx = \frac{1}{200} = 0.005$$

$$1.19753 + 0.004132 = 1.201662 \leq \sum_{n=1}^{\infty} \frac{1}{n^3} \leq 1.20253 = 1.19753 + .005$$

The lower and upper bounds are less than 0.001 apart.