Monday Night Calculus

The Fundamental Theorem of Calculus

11/30 Question

1. Suppose the function g is defined by $g(x) = \int_2^x 2^{-t^2}\,dt$.

(a) Find an equation of the line tangent to the graph of g at $x = 2$.

$$g(2) = \int_2^2 2^{-t^2}\,dt = 0$$

$$g'(x) = 2^{-x^2} \quad \Rightarrow \quad g'(2) = 2^{-4} = \frac{1}{16}$$

An equation of the tangent line: $y = \frac{1}{16}(x-2)$

(b) Let $h(x) = g(\sqrt{x})$. Find $h(4)$, $h'(4)$, and $h''(4)$.

$$h(4) = g(\sqrt{4}) = g(2) = 0$$

$$h'(x) = g'(\sqrt{x}) \cdot \frac{1}{2}x^{-1/2} = \frac{2^{-x}}{2\sqrt{x}} \quad \Rightarrow \quad h'(4) = \frac{2^{-4}}{2 \cdot 2} = \frac{1}{64}$$

$$h''(x) = \frac{1}{2}\left[\frac{\sqrt{x} \cdot 2^{-x}(-1)\ln 2 - 2^{-x}\frac{1}{2}x^{-1/2}}{x}\right]$$

$$= \cdots = \frac{2^{-x}}{4}\left[\frac{-2\ln 2 \cdot x - 1}{x^{3/2}}\right]$$

$$h''(4) = \frac{2^{-4}}{4}\left[\frac{-2\ln 2 \cdot 4 - 1}{8}\right] = \frac{1}{512}(-8\ln 2 - 1)$$

2. Let k be the function defined by $k(x) = \int_{\sin x}^{\cos x} t\,dt$.

(a) Find $k(0)$ and $k\left(\frac{\pi}{4}\right)$.

$$k(0) = \int_{\sin 0}^{\cos 0} t\,dt = \int_0^1 t\,dt$$
$$= \left[\frac{t^2}{2}\right]_0^1 = \frac{1}{2} - \frac{0}{2} = \frac{1}{2}$$

$$k\left(\frac{\pi}{4}\right) = \int_{\sin(\pi/4)}^{\cos(\pi/4)} t\,dt = \int_{\sqrt{2}/2}^{\sqrt{2}/2} t\,dt = 0$$

(b) Use the Fundamental Theorem of Calculus Part 1 to find $k'(x)$.

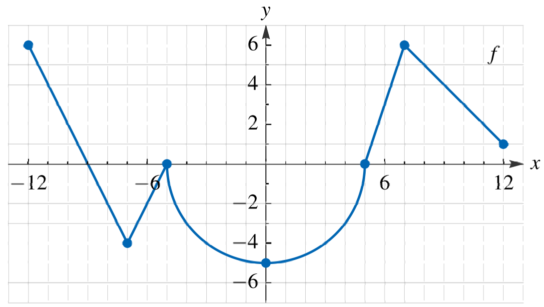
$$k(x) = \int_{\sin x}^{0} t\,dt + \int_0^{\cos x} t\,dt$$
$$= -\int_0^{\sin x} t\,dt + \int_0^{\cos x} t\,dt$$

$$k'(x) = -(\sin x)(\cos x) + (\cos x)(-\sin x)$$
$$= -2\sin x \cos x$$

(c) Use the Fundamental Theorem of Calculus Part 2 to show that $k(x) = \frac{1}{2}\cos(2x)$.

$$k(x) = \left[\frac{t^2}{2}\right]_{\sin x}^{\cos x}$$
$$= \frac{1}{2}[\cos^2 x - \sin^2 x] = \frac{1}{2}\cos(2x)$$

3. Let g be the function defined by $g(x) = \int_5^x f(t)\,dt$ where f is the function whose graph is shown in the figure.

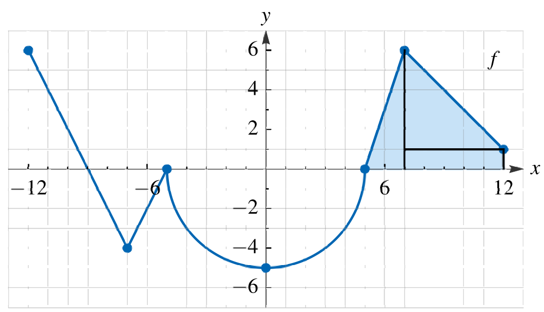


(a) Find $g(12)$ and $g(-5)$.

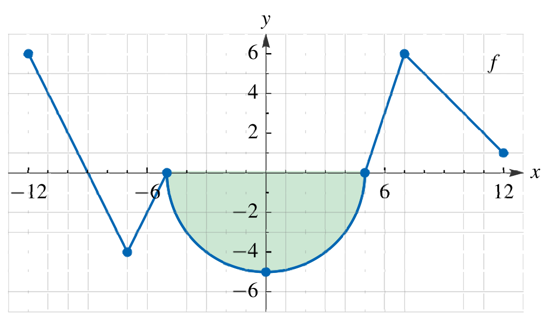
$$g(12) = \int_5^{12} f(t)\,dt = \int_5^7 f(t)\,dt + \int_7^{12} f(t)\,dt$$

$$= \frac{1}{2}\cdot 2\cdot 6 + \frac{1}{2}\cdot 5\cdot 5 + 5$$

$$= 6 + \frac{25}{2} + 5 = \frac{47}{2}$$



$$g(-5) = \int_5^{-5} f(t)\,dt = -\int_{-5}^{5} f(t)\,dt = -\left(-\frac{1}{2}\pi 5^2\right) = \frac{25\pi}{2}$$



(b) Find the maximum value of g on the closed interval $[-12, 12]$.

$g'(x) = f(x)$

$g'(x) = 0 : x = -9,\ -5,\ 5$

$g'(x)$ DNE : none

Eliminate $x = -5$ and $x = 5$

$$g(-9) = \int_5^{-9} f(t)\,dt = -\int_{-9}^{5} f(t)\,dt = -\left(-\frac{1}{2}\cdot 4\cdot 4 - \frac{25\pi}{2}\right) = 8 + \frac{25\pi}{2}$$

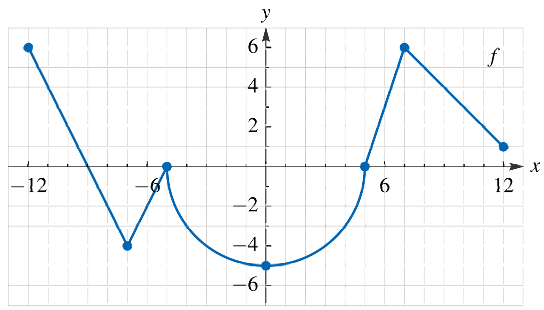
$$g(-12) = \int_5^{-12} f(t)\,dt = -\int_{-12}^{5} f(t)\,dt = -\left[\int_{-12}^{-9} f(t)\,dt + \int_{-9}^{5} f(t)\,dt\right]$$

$$= -\left[\frac{1}{2}\cdot 3\cdot 6 + \left(-8 - \frac{25\pi}{2}\right)\right]$$

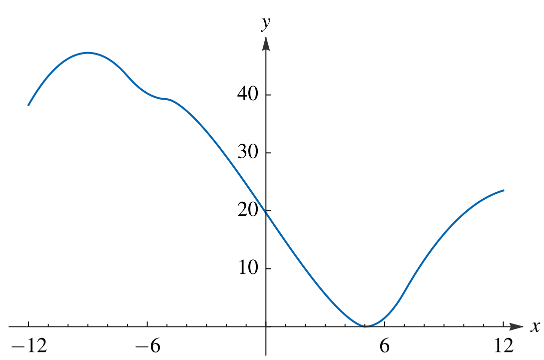
$$= -\left[9 - 8 - \frac{25\pi}{2}\right] = \frac{25\pi}{2} - 1$$

$$g(12) = \frac{47}{2}$$

The absolute maximum value of g on the interval $[-12, 12]$ is $g(-9) = 8 + \dfrac{25\pi}{2}$



Here is a graph of g.



(c) Find an equation of the line tangent to the graph of g at $x = 7$, or explain why the tangent line does not exist.

$$g(7) = \int_5^7 f(t)\,dt = \frac{1}{2} \cdot 2 \cdot 6 = 6$$

$$g'(x) = f(x) \quad \Rightarrow \quad g'(7) = f(7) = 6$$

An equation of the tangent line: $y - 6 = 6(x - 7) \quad \Rightarrow \quad y = 6x - 36$

(d) Let $h(x) = g(x^2)$. Find $h'(-3)$.

$$h'(x) = g'(x^2) \cdot 2x$$

$$h'(-3) = g'(9) \cdot (-6) = f(9)(-6) = (4)(-6) = -24$$

(e) Let $k(x) = g(x)^2$. Find $k'(9)$.

$$k'(x) = 2g(x)g'(x)$$

$$k'(9) = 2g(9)g'(9) = 2\left[6 + \frac{1}{2}(2)(6 + 4)\right] f(9) = 2 \cdot 16 \cdot 4 = 128$$