

# Approximating Gradient and Area



## Student Activity Sheet

7 8 9 10 11 12



## Introduction

Differential and Integral Calculus in VCE Mathematical Methods U3&4 from 2023 onwards requires students to be familiar with techniques for approximating seemingly impossible calculations. These are the gradient of the tangent to a curve (What? Gradient requires two points, but a tangent line has only one point in common with the curve...!!) and area between a curve and the x-axis (Hold on!! How can you do that if the curve is a parabola, or a cubic? I don't know any area formulae for these shapes!!).

These queries are precisely why we need to have a way of getting *close* to the required calculation value. By learning how TI-Nspire CAS technology can assist us in this process, we should have a better appreciation for the power of Differential and Integral Calculus to find the *exact* gradient and area values.

## Central Difference Approximation – some theory

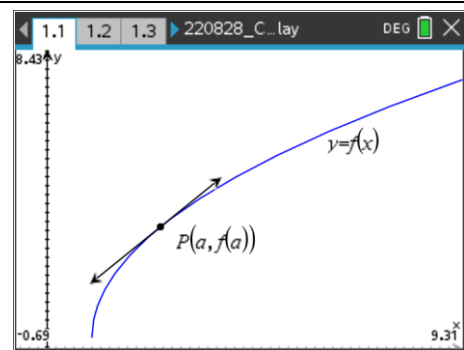
### The problem:

Find the gradient of the tangent line to a graph of a function as shown opposite.

### Note:

In assessment questions, you will work with a specific function (here, called  $f(x)$ ) and a specific x-coordinate of P (here, called  $a$ ).

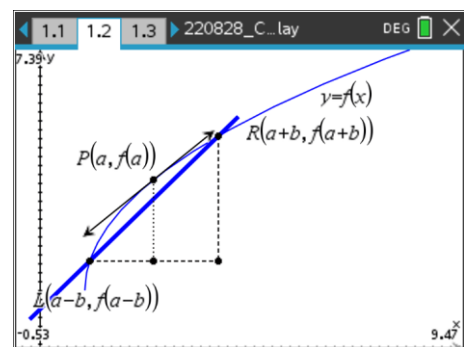
$P(a, f(a))$  is the point of tangency.



### The task:

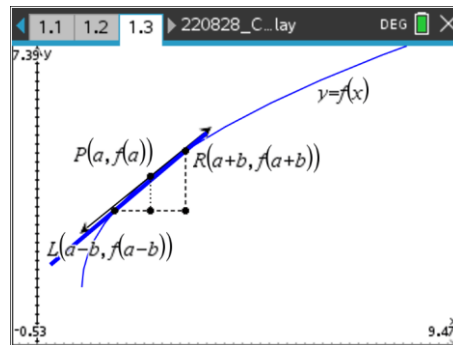
As gradient of a line requires two specific points, the Central Difference technique asks you to choose two points L (for **left** of P) and R (for **right** of P), nearby P on the graph of  $f(x)$  which are **equidistant** (horizontally) from P. Thus you need to find (or, you'll be told to use) a suitable value for  $b$  which will be used to find the x-coordinates of L ( $a-b, f(a-b)$ ) and R ( $a+b, f(a+b)$ ).

The blue line,  $\overline{LR}$  is a secant line as it intersects the graph of  $y=f(x)$  twice. Here, note its gradient is close to that of the tangent line – they're not quite parallel.



**An important observation:**

This diagram shows the same graph of  $y = f(x)$ , but the value of  $b$  is now smaller than it was in the previous diagram; so points L and R are closer to P, and the secant line's gradient appears to be much closer to that of the tangent line.



In this work, the gradient of the tangent line is written as  $f'(x)$ . We use the familiar gradient formula:

$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{with points L and R to write:} \quad f'(x) \approx \frac{f(a+b) - f(a-b)}{(a+b) - (a-b)} \approx \frac{f(a+b) - f(a-b)}{2b}$$

**Question: 1.**

As the value of  $b$  becomes \_\_\_\_\_, the gradient of the secant becomes \_\_\_\_\_.

**Question: 2.**

Suppose you are asked to find the approximate value of the gradient to the curve:  $f(x) = (x-2)\sqrt{x+1}$  at P where  $x = 2$ , using decreasing  $b$ -values 1 and 0.5. Complete the table below:

$f(x) = (x-2)\sqrt{x+1}$ and $a = 2$					
$b$	$a+b$	$f(a+b)$	$a-b$	$f(a-b)$	$f'(x) \approx \frac{f(a+b) - f(a-b)}{2b}$
1					
0.5					

**Question: 3.**

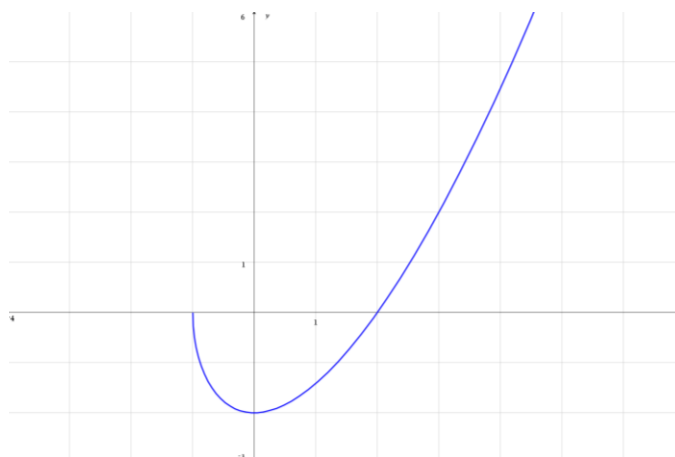
The graph of the function  $f(x) = (x-2)\sqrt{x+1}$  is shown opposite.

Locate point P where  $a = 2$  and also the two points L and R for  $b = 1$ .

Use a ruler to sketch the tangent at P in black and the secant  $\overline{LR}$  in red.

It is known that the gradient of the tangent at P is exactly  $\sqrt{3}$ .

Comment on the link between your sketch of the secant and the value you found using the Central Difference approximation technique.



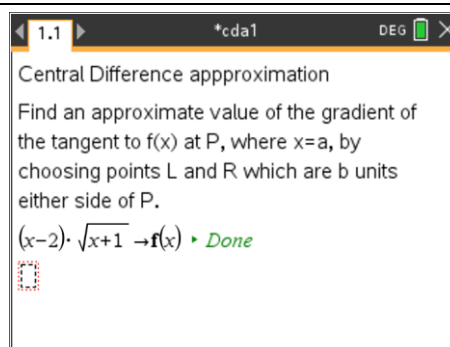
## Central Difference Approximation – Using Nspire

We're now going to see how the calculations for secant gradients can be easily done on Nspire. We'll use the same function  $f(x) = (x-2)\sqrt{x+1}$  with Point P located on the graph at  $x = 2$ . To do this we can use either a Notes page or a Lists & Spreadsheet page; shown below is the Notes page approach.

Open the file *cda1.tns* whose p1.1 is a Notes page.

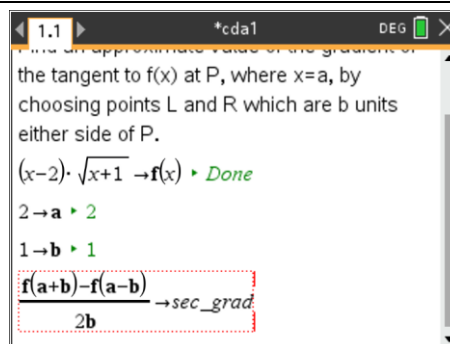
Using **ctrl M** to open a Math Box, key in the function rule:

$f(x) = (x-2)\sqrt{x+1}$  then press **ctrl** and **var** then  $f(x)$  to store this function; press **enter** and a second maths box will appear. If the second maths box doesn't appear, press **ctrl M**.



In the same way, store  $2 \rightarrow a$  and for now, store  $1 \rightarrow b$ . Press **enter** after each. Now key in the formula for the secant gradient

$\frac{f(a+b) - f(a-b)}{2b}$  and store it as the variable *sec\_grad*.



### Question: 4.

When you press **enter** now, what do you notice?

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### Question: 5.

When you go into the Math Box for the variable  $b$  and change it to 0.5, what do you notice?

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### Question: 6.

Choose two more successively smaller values of  $b$  for your Notes page and comment.

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### Question: 7.

(Extending) Now vary stored values of  $a$  or  $b$ , and/or for the function  $f(x)$

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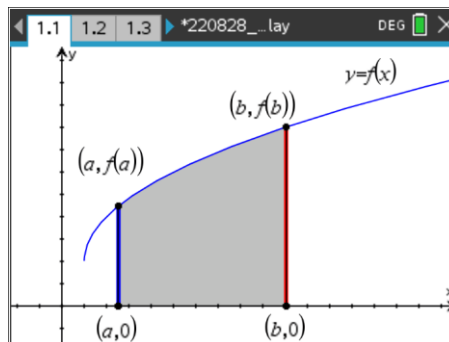


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## Trapezium Rule Approximation for Area – some theory

### The problem:

Find the area between a curve defined as  $y = f(x)$  and the  $x$ -axis and two vertical boundaries  $x = a$  (in blue) and  $x = b$  (in red), but we have no area formula for this shape.

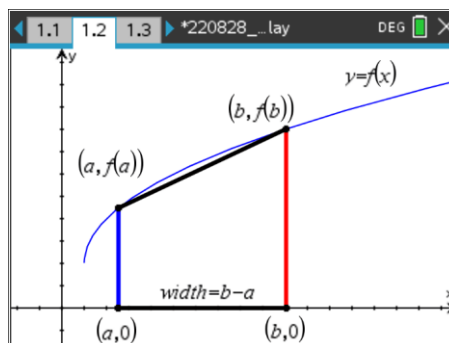


We can get close to this area by forming a trapezium. In most textbooks, that formula appears as:

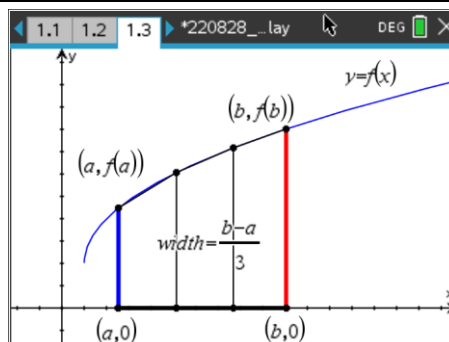
$$A = \frac{h(a+b)}{2}$$

It involves multiplying the *average* of the parallel side lengths (usually drawn horizontally) by the *height* (distance between those parallels). In our diagrams, the parallels are vertical; the blue with length  $f(a)$  and the red with length  $f(b)$ , and we'll replace the previous "height" with the *width* which is  $b - a$ .

This trapezium area is approximately equal to the required area shaded in the previous graph.



An even better approximation will be found if we form more than one trapezium between the boundaries. The width of each trapezium must be the same, and will be some fraction of the original  $b - a$ . You can see there's less "wastage" of area here, as the trapezia are much closer to the curve.



For Questions 8 and 9 below, you may wish to use your Nspire CAS to sketch the graph and perform calculations)

### Question: 8.

Consider the curve given by  $f(x) = (x - 3)^2 + 2$ . Suppose the approximate bounded area between the graph of  $f(x)$ , the  $x$ -axis and the lines  $x = 2$  and  $x = 5$  is required.

Find the width if 1 trapezium is used; 3 trapezia? 6 trapezia?

Area using 1 Trapezium:

Area using 3 Trapezia:

Area using 6 Trapezia:

**Question: 9.**

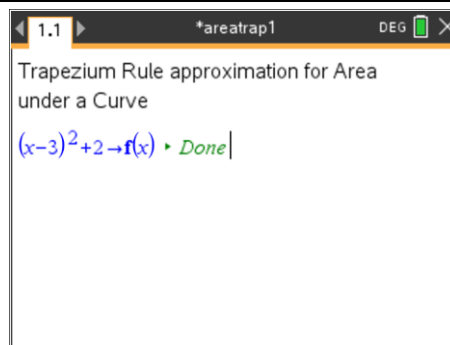
For the approximate area described in Question 8 above, the lengths of the left and right boundaries of the various trapezia need to be calculated. Suppose we're looking at using 3 trapezia between the lines  $x = 2$  and  $x = 5$ . The first trapezium is between  $x = 2$  and  $x = 3$ , so its left boundary has length  $f(2) = 3$ , and its right boundary is  $f(3) = 2$ . That trapezium has an area of  $\frac{1 \times (3+2)}{2}$ , or 2.5 sq units. Find the areas of the other two trapezia, and therefore the total approximate area.

**Trapezium Rule Approximation for Area – Using Nspire**

We're now going to see how the calculations for trapezium areas can be easily done on Nspire. We'll use the same function  $f(x) = (x - 3)^2 + 2$  with vertical boundaries located at  $x = 2$  and  $x = 5$ . To do this we can use either a Notes application or a Lists & Spreadsheet application; shown below is the Notes application approach.

Open the file *areatrap1.tns* whose p1.1 is a Notes application.

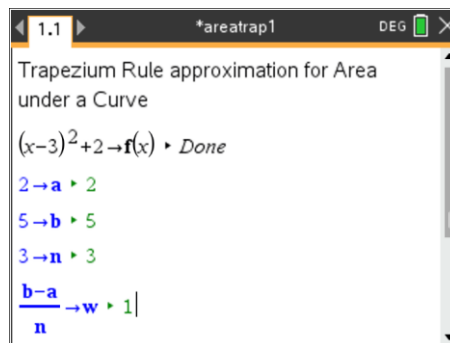
As seen previously, press **ctrl M** to activate the Maths Box; into this key in the function rule  $(x - 3)^2 + 2$ , store it using **ctrl var** as  $f(x)$ , and press **enter**.



Similarly, store the values of  $a = 2$  and  $b = 5$  as shown.

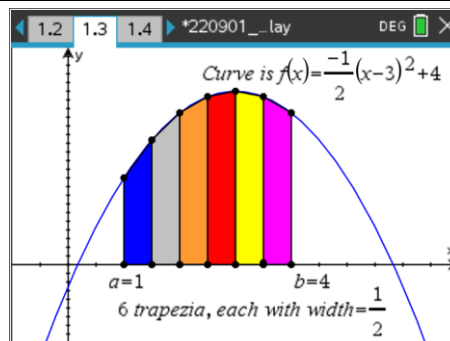
We use  $n$  as the number of trapezia we wish to use; here, let's start with  $n = 3$ . Store that as well.

Since width (the fraction of the original  $b - a$ ) is a crucial value for the area calculation, and referring to Q8, store  $\frac{b - a}{n}$  as the variable  $w$ ; press **enter**, this will allow the width to change as you change the values of  $a$ ,  $b$  and  $n$ .



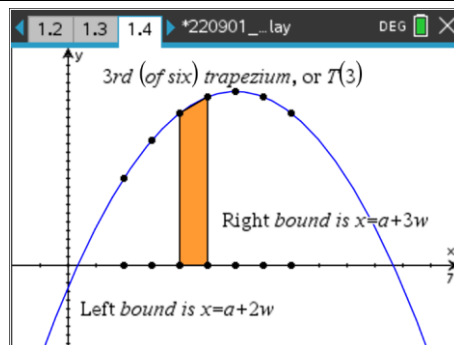
Before we get to the major formula, it's worth noting the diagram on the right for another quadratic function whose required area has been divided into 6 equally-spaced trapezia.

Notice here that  $a = 1$ ,  $b = 4$  and  $n = 6$ , so  $w = \frac{4-1}{6} = \frac{1}{2}$ .



Let's look at the 3<sup>rd</sup> trapezium, called T(3). The x-coordinates of its vertical boundaries are  $a + 2w$  and  $a + 3w$ , so the lengths of these boundaries are  $f(a + 2w)$  and  $f(a + 3w)$ , respectively.

The area of T(3) is therefore  $\frac{1}{2} \times \frac{f(a + 2w) + f(a + 3w)}{2}$ ; the others can be expressed in a similar format.



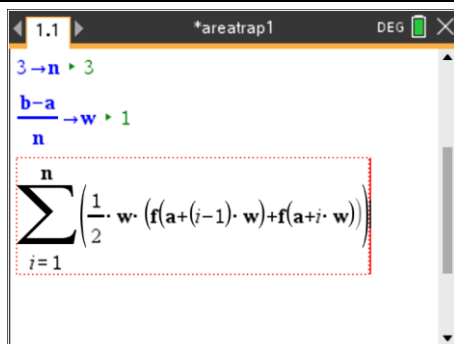
Now we're going to use Notes to store the formula for using these values to sum (or, add up) all the trapezia areas at once. The formula is:

$$\sum_{i=1}^n w \times \frac{f(a + (i-1) \times w) + f(a + i \times w)}{2}$$

Looks messy, but the summation sign indicates the values of  $i$  (1,2,3,4,5,6) are substituted into the formula, one by one, and the calculations are then added to produce the overall approximate area.

We go back to the Notes p1.1; with the cursor in a Math Box we press the template key:  $\sum_{i=1}^n$  (found to the right of the 9 key) and find the summation template. Carefully fill in details as shown – use multiplication signs where required.

When you press enter, the required area sum will be displayed.



**Question: 10.**

How does your answer compare to that found in Q9 earlier?

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**Question: 11.**

Now suppose we want to look at the approximate area with 6 trapezia, not 3. Alter the Math Box for the variable  $n$ , and discuss the meaning of your result compared with that found in Question 10. Use the fact that the actual exact area, in this case, is a whole number.

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