Geometric Sequences & Series

Teachers Teaching with Technology™
Professional Development from Texas Instruments

Answers

7 8 9 10 **11** 12

TI-Nspire™ Investigation Student 30 min

Introduction

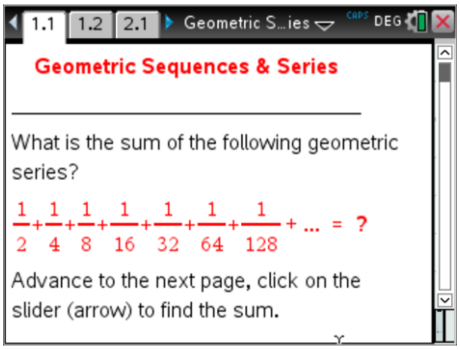
Imagine you just arrived home from school, you're really hungry. You decide the block of chocolate will help you with your homework. Pretty soon half the block is gone, so you quickly put it back in the refrigerator. Your brother arrives home, spots the half eaten block of chocolate and immediately breaks half the remaining block off for himself. Dad's next, when he visits the refrigerator he too breaks off half the remaining block. How long will the block last you wonder?

In practice, the block would probably not last much longer; the next visitor to the refrigerator would probably eat the remaining portion, however in some situations this diminishing process can last for a very long time, in mathematics we also imagine it going on forever, to infinity. When a set of numbers is either diminishing, or increase at a constant rate they form a Geometric Sequence; the sum of these numbers is referred to as a Series.

Instructions

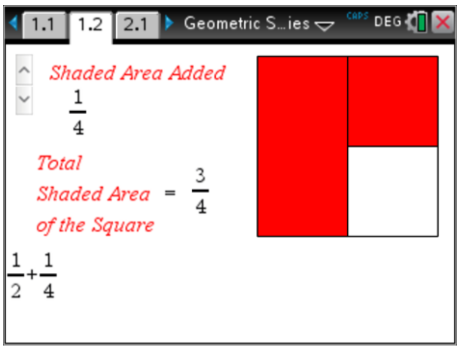
Open the TI-nspire file "Geometric Sequences and Series".

The fractions on the first page relate to the diminishing block of chocolate. When the problem is put into a pure mathematical equation the answer is perhaps less obvious.



Navigate to page 1.2.

Use the slider (top right) to see how the progressive sum of each term appears visually and numerically.



Question: 1.

What is the answer to the infinite sum: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}\ldots$ Based on the image: 1

How would our problem change if only a third of the remaining block was consumed each time a family member came along? The amount each member consumes is still diminishing, this time the factor[1] is: $\frac{1}{3}$ instead of: $\frac{1}{2}$.

[1] Factor – This is also referred to as the common ratio.

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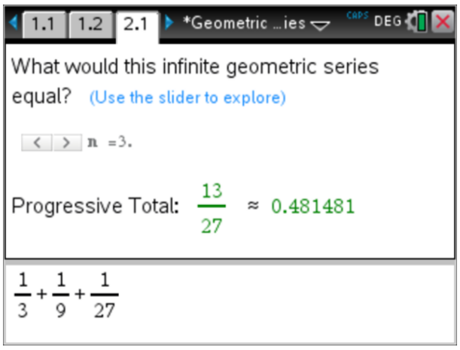
Navigate to page 2.1.

Use the slider (top right) to explore the progressive sum of this sequence.



Question: 2.

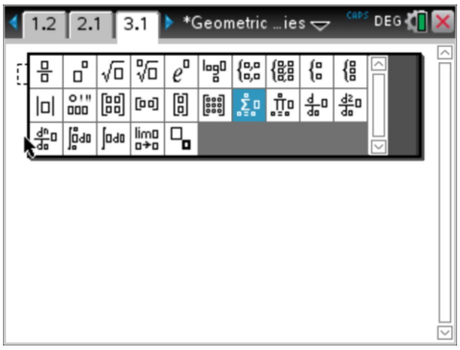
What is the answer to the infinite sum: $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\frac{1}{243}...$ Based on the approximate answer: ½

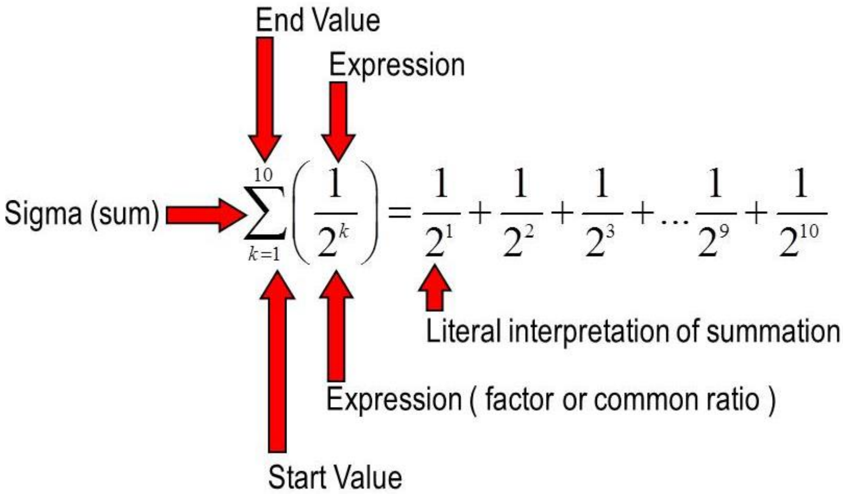
Navigate to page 3.1.

The calculator has a summation command that can be used to determine the sum of a sequence. The Greek symbol sigma is used for computing the sum of a set of numbers connected by a formula.

Press the template key to reveal a collection of mathematics templates, navigate to the summation symbol.

Enter the expression shown below, including the start and end values.





Question: 3.

Determine each of the following sums:

a. $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}+\frac{1}{256}$ $(2^8 = 256)$ $= \frac{255}{256}$

b. $\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}+\frac{1}{256}+\frac{1}{512}+\frac{1}{1024}+\frac{1}{2048}$ $(2^{11} = 2048) =$ $= \frac{511}{2048}$

c. $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\frac{1}{243}+\frac{1}{729}+\frac{1}{2187}+\frac{1}{6561}+\frac{1}{19683}$ $(3^9 = 19683)$ $= \frac{9841}{19683}$

d. $\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\frac{1}{10000}+\frac{1}{100000}+\frac{1}{1000000}+\frac{1}{10000000}$ $= \frac{111111}{10000000}$

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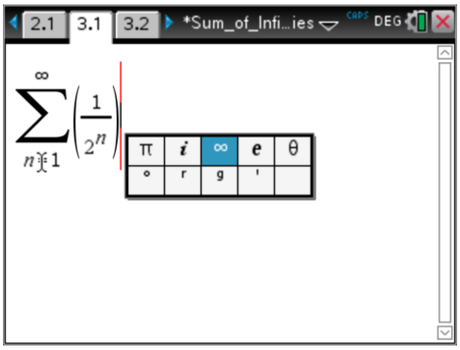
The original problem involved summing terms 'forever', an infinite number of terms. The summation formula can handle this option.

The infinity symbol can be obtained from the common symbols and constants: [π▸]



Question: 4.

Determine each of the following infinite sums:

a. $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}\ldots$

Compare your answer with the one estimated in Question 1.

b. $\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}\ldots = 1-\frac{1}{2}-\frac{1}{4}=\frac{1}{4}$

Explain how else the calculation could be determined using your result from part (a)

c. $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\frac{1}{243}\ldots = \frac{1}{2}$

Compare your answer with the one estimated in Question 2.

d. $\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\frac{1}{10000}+\frac{1}{100000}\ldots = \frac{1}{9}$

Generating a Formula

Sometimes it is quicker to use a specific formula rather than the summation command. Consider the following sequence:

Term Number	1	2	3	4	…	n
Term	a	$a \times r$	$a \times r \times r$	$a \times r \times r \times r$		$a \times r^{n-1}$

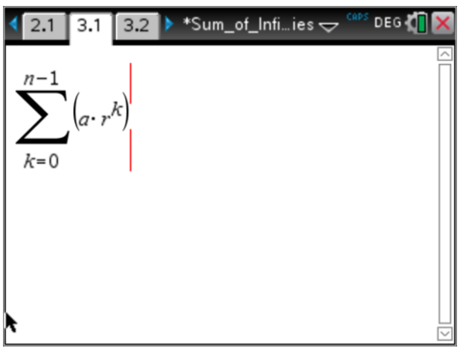
In this sequence (above), a is the first term, r is the common ratio and n is the number of terms in the sequence.

The TI-Nspire CX CAS is capable of generating formulas given the appropriate information.

Enter the expression:

$$\sum_{k=0}^{n-1} a \cdot r^k$$

Once the calculator has produced an answer, use the Algebra menu and select the Factor command and factorise the answer.



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Question: 5.

Write down the formula generated by the calculator and explain the following: $\dfrac{a(r^n-1)}{r-1}$

a. Why does the summation go to n – 1 for the first n terms?

The first term is a, the second term is: $a \cdot r^1$, the third term is $a \cdot r^2$ so the power is always one less than the number of terms. The first term does is not multiplied by the common ratio so there are only n – 1 that are multiplied by the ratio.

b. Why does the summation start with k = 0?

The first term in the summation could be thought of as $a \cdot r^0$, alternatively, the second term is $a \cdot r^1$ so the second term needs to have k = 1 which means the first term needs to have k = 0.

Question: 6.

Consider the sequence: $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}$

a. What is the first term? (a) a = 1

b. What is the common ratio? (r) r = ½

c. How many terms are there in the sequence? (n) n = 7

d. Determine the sum of the terms in this sequence using the formula generated for Question 5.

Using the formula: $\dfrac{1\times\left(\frac{1}{2^7}-1\right)}{\frac{1}{2}-1}=\dfrac{127}{64}$.

The same result can be used using the summation command: $\sum_{k=0}^{6} 1\cdot\dfrac{1}{2^k}$

Chessboard Problem

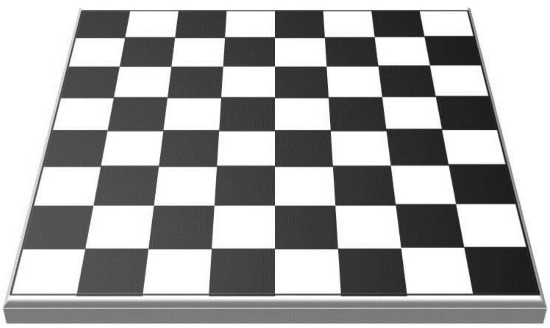
So far all the sequences have been diminishing. Whilst it might be nice to have a chocolate block that grows and grows, this problem is about a Chessboard. A mythical story about the creation of the game of Chess states that the King offered its creator a huge sum of money, instead the creator asked for the following:

"I would like some rice to feed my family. I would like one grain of rice on the first square of the chessboard, two grains of rice on the second, four on the next, then eight and so on… all the way to the 64th square.

Foolishly the King agreed to this request.

Question: 7.

How many grains of rice should the creator of Chess receive? Consider how much it would weigh, the amount of space it would occupy and how long the creator could feed his family.



$$\sum_{k=0}^{63} 2^k = 18{,}446{,}744{,}073{,}709{,}551{,}615$$

It is possible to calculate the weight of an average grain of rice, the amount is approximately 0.03grams.

Weight: Approximately 553,402,322,211 tons!

Volume: 368,934,881,474m^3 … that's enough to fill 210,819,932 Olympic sized swimming pools.

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Teacher Notes

There are many opportunities to extend this activity such as determining the formula for an infinite series. Students can investigate what happens to r^n (where $|r|<1$) and therefore deduce the formula. This is a much better long term solution as the formula will then have meaning.

Recurring decimals can also be aligned with ones similar to Question 4(d).

$$\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\frac{1}{10000}+\frac{1}{100000}...$$

For example: 0.22222… is equivalent to 0.2 + 0.02 + 0.002 + 0.0002 + 0.00002. This can be written as a fraction in the form:

$$\frac{2}{10}+\frac{2}{100}+\frac{2}{1000}+\frac{2}{10000}+\frac{2}{100000}...$$

Or less trivial variations:

$$\frac{27}{100}+\frac{27}{10,000}+\frac{27}{1,000,000}+\frac{27}{100,000,000}+...=\sum_{k=1}^{n}\left(27\frac{1}{100^2}\right)$$

Check out the other activities on the Texas Instruments Australian website – Senior Curriculum Inspirations, including problem solving such as “Pebbling the Chessboard” and using sequences and series using recursive formulas, spreadsheets and graphs.

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