Vector Graphics

Students

Introduction

In 1637 René Descartes published a revolutionary book La Géométrie where he brought together two areas of mathematics; Geometry and Algebra. The "Cartesian" plane, named after René Descartes, is an integral component of every high school mathematics course. In this activity you will see how vectors can be incorporated into this two dimensional space. Using the same techniques, you can easily extend this approach into three dimensions.

Vectors can be expressed as a direction and magnitude (Polar) or as mutually perpendicular projections (Components), the latter is the focus of this activity. The directions are referred to as i (x direction) and j (y direction), for three dimensions a k (z direction) is also included. For some relatively straight forward problems it is quicker and more appropriate to complete calculations 'by-hand', however, as the calculations are repeated and gradually increase in complexity; having a template becomes an extremely useful tool. This activity sets up a template that can be used for other investigations including problems in three dimensions.
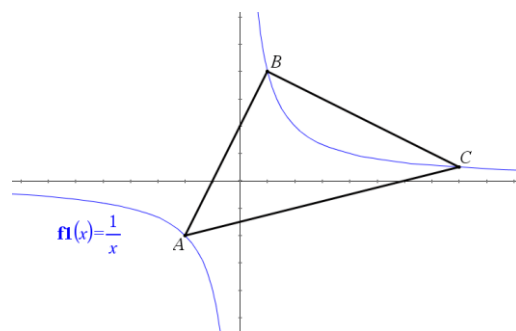
Problem to be explored

Consider three points A, B and C on the function: $y = \dfrac{1}{x}$.

These three points can be joined to create a triangle: $\triangle ABC$.

This triangle has some interesting features that will be explored through the application of vectors.

To begin this investigation, point A = (-1, -1), B = (½ , 2) and C = (4, ¼).

Vectors as Matrices

A vector in **component** form can be used to reference point A:

$$OA = \underset{\sim}{a} = -\underset{\sim}{i} - \underset{\sim}{j}$$

One way to represent a vector is to use a matrix. This vector can be stored in the Calculator or the Notes application for a more dynamic and responsive experience.

In a new document, insert a Notes application and insert a Maths box to define matrix OA.

Use the template tool to insert a 1×2 matrix to represent vector OA.

Use the above process to also create vectors OB and OC.
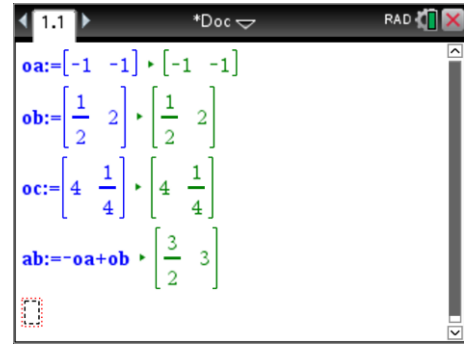
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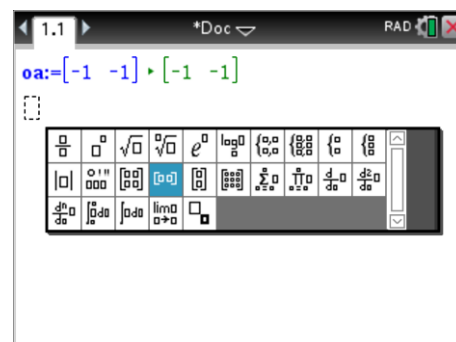
Vector AB is equal to: AO + OB or ⁻OA + OB

Define vector AB as –OA + OB, this will allow vector AB to update automatically as OA or OB is changed.

Note:

    The calculator does not recognise AO as the negative of OA.

Question 1

Define expressions for vectors BC and CA in terms of OA, OB and OC as applicable and write down the matrix and vector representations.

Question 2

In a Calculator application, calculate: AB + BC + CA and explain the result.
Note: [VAR] key can be used to quickly access and recall each defined vector.

Question 3

Draw a diagram of the graph and all six vectors. (Label each vector)

Magnitude of a Vector

The magnitude of vector is calculated using Pythagoras's theorem, squaring each component, adding these squared values together and then calculating the square-root:

The magnitude of a vector can be computed by:

$$|\underset{\sim}{a}| = \sqrt{x^2 + y^2} \quad \text{where} \quad \underset{\sim}{a} = x\underset{\sim}{i} + y\underset{\sim}{j}$$

The "Norm" command within the Matrix and Vector menu computes the magnitude of a vector.
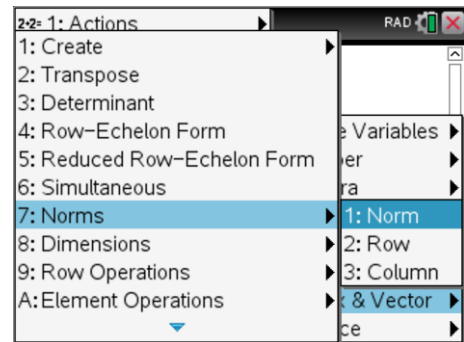
We can see how this works with matrices step by step:

$$\underset{\sim}{a} = \begin{bmatrix} x & y \end{bmatrix}$$

To multiply matrix A by itself requires the matrix to be transposed so the rows and columns align:

$$\left(\underset{\sim}{a}\right)^T = \begin{bmatrix} x \\ y \end{bmatrix}$$

The product is therefore:

$$\underset{\sim}{a} \times \underset{\sim}{a}^T = \begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x^2 + y^2 \end{bmatrix}$$

The determinant of this 1 x 1 matrix is simply $x^2 + y^2$ which then allows the square-root of this value to be computed.
All of this is achieved when the "Norm" command is used.

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Mathematics Tip!

In "vector world", the magnitude of a vector can also be computed using the dot product:

$$|\underset{\sim}{a}| = \sqrt{\underset{\sim}{a} \cdot \underset{\sim}{a}}$$

Question 4

Determine the magnitude of vectors AB, BC and CA and hence the perimeter of triangle ABC.

Calculator Tip!

The basis for the entire problem has been set up as a template in a Notes application. The template can be duplicated and used in new problems.

- Navigate to the document view (Ctrl + Up)
- Copy the Notes application (Ctrl + C)
- Use the Doc (document) key to insert a **New Problem**
- Paste the Notes application (Ctrl + V)

Suppose point A (-1, -1) is moved to a new location on the curve $y = \dfrac{1}{x}$, the perimeter of the triangle would obviously change.

The coordinates of point A could be defined as: $\left(x, \dfrac{1}{x} \right)$

Create a duplicate of the original Notes application (template) and paste this into a new problem. (See calculator tip above). Notice that the page label changes to 2.1 to indicate Page 1 of Problem 2.

To use variables in a vector (matrix) on the TI-Nspire CX (non-CAS) the variable must be defined as a function, as shown opposite, this includes the addition of the vectors for AB, BC and CX.

Redefine each of the vectors as a function. Note that error messages may be displayed during this process as all other dependent vectors require functional definitions.

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Suppose point A is moved such that the perimeter is now equal to 17. This means that the sum of the side lengths, vector magnitudes, will be equal to 17.

$$|AB|+|BC|+|CA|=17$$

The equation can be solved using the numerical solver (as shown).
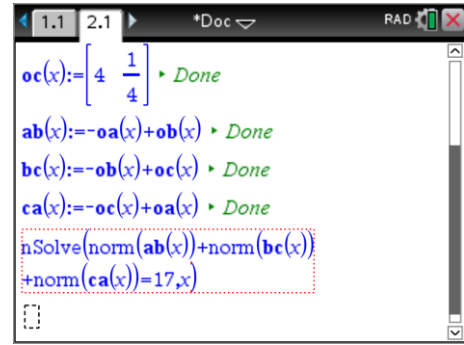
Calculator Tip!

We often find in mathematics that there is more than one solution to a problem. When using the numerical solver command (nSolve), domain constraints can be used to help locate specific solutions. For example the following command will only look for solutions between -1 and 0.

nSolve(norm(ab(x)) + norm(bc) + norm(ca(x)) = 17, x) | -1 < x < 0

Question 5

What location(s) are possible for point A to generate a perimeter of 17?

Dot Product of a Vector

The dot product of two vectors is computed as follows:

$$\underset{\sim}{a}\cdot\underset{\sim}{b} = x_a x_b + y_a y_b + z_a z_b \text{ where: } \underset{\sim}{a} = x_a \underset{\sim}{i} + y_a \underset{\sim}{j} + z_a \underset{\sim}{k} \text{ and } \underset{\sim}{b} = x_b \underset{\sim}{i} + y_b \underset{\sim}{j} + z_b \underset{\sim}{k}$$

$$\underset{\sim}{a}\cdot\underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos(\theta) \text{ where } \theta \text{ is the angle between the two vectors.}$$

The dot product of two vectors can be used to compute the angle between the vectors, in this section however we will use the fact that if the dot product is equal to zero then the vectors are at right angles. [cos(90°)=0]

It is easy to calculate the dot product of two vectors via the matrix and vector menu:

[Menu] > Calculations > Matrix & Vector > Vector > Dot Product
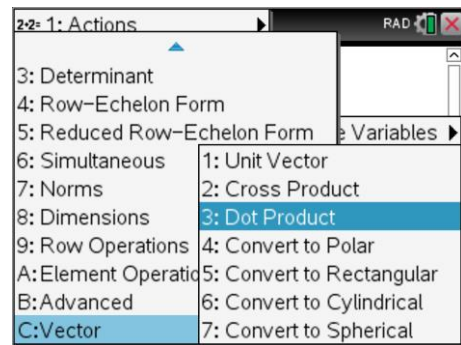
Question 6

Navigate back to the original template (Page 1.1). Change where location of point A is: (-1, -1). Calculate the dot product of vectors AB and BC.
What does the result say about the angle between these two sides?

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Question 7

Using your answer to question 6 to help determine the area of the triangle ABC where A is located at (-1, -1).

Suppose the vertices of the triangle are now:

A = (-1, 1)   B = $\left(\frac{1}{3}, 3\right)$ C = $\left(c, \frac{1}{c}\right)$

Question 8

Point C is to be moved such that a right angled triangle is to be formed once again.

a)   Determine algebraic expressions for vectors BC and CA.

b)   Determine an algebraic equation for the dot product of vector BC and CA given BC $\perp$ CA.

c)   What is/are the value(s) of C for which ABC is once again a right angled triangle?
Note: The result can be expressed as an exact value.

d)   Use the numerical solve to find solutions to: nSolve(dotP(bc(x),ca(x))=0,x)
Note: Domain constraints will need to be applied to x to locate all of the solutions.

Question 9

Verify that the value of C forms a right angled triangle.

Vector multiplied by a Scalar

When a vector is multiplied by a scalar, the direction[1] of the vector remains the same but the length will change if the scalar quantity is not equal to 1. A simple way to explore this is through a geometric example.
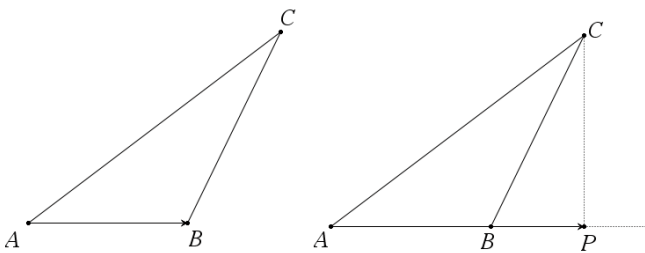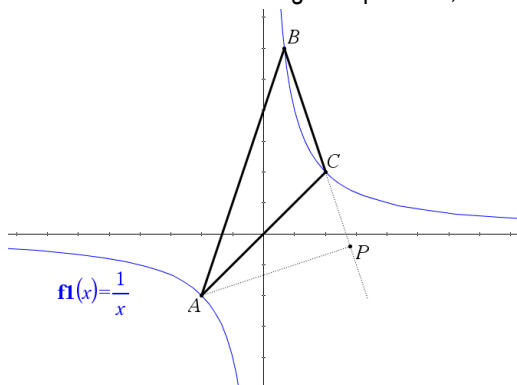

Consider the triangles shown here. When trying to calculate the height of $\triangle$ABC we can extend the base to point P such that $\angle$APC = 90°.

Vector AB can be multiplied by a scalar n such that vector AP = nAB.

Vector AP has the same direction as AB but has a different length. **Note**: PC is the altitude of $\triangle$ABC.

For this section of the investigation points A, B and C are defined as:

A = (-1, -1)   B = $\left(\frac{1}{3}, 3\right)$ C = (1, 1)

Notice that BC has been extended such that BP = nBC.

P is located such that PA is perpendicular to BP.

This means that PA·BP = 0

Note: AP is the altitude of $\triangle$ABC.

Question 10
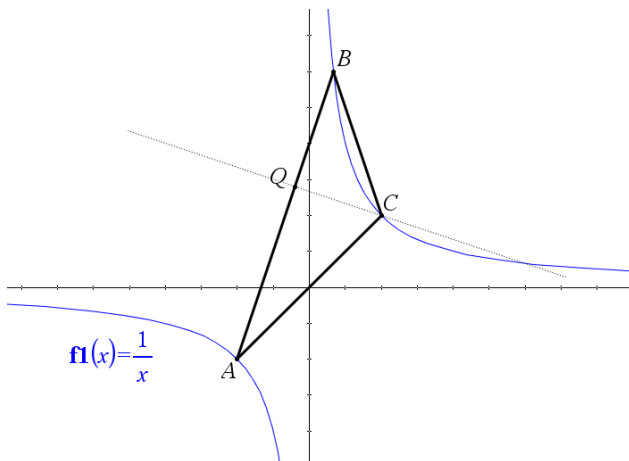
Write a vector equation for AP in terms of BC and AB.

Question 11

Determine the value of n such that AP·BP = 0

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[1] Direction – The direction will reverse if the scalar is a negative quantity.

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Question 12

Determine the vectors AP, OP and the coordinates of point P.

A = (-1, -1)

B = $\left(\frac{1}{3}, 3\right)$

C = (1, 1)
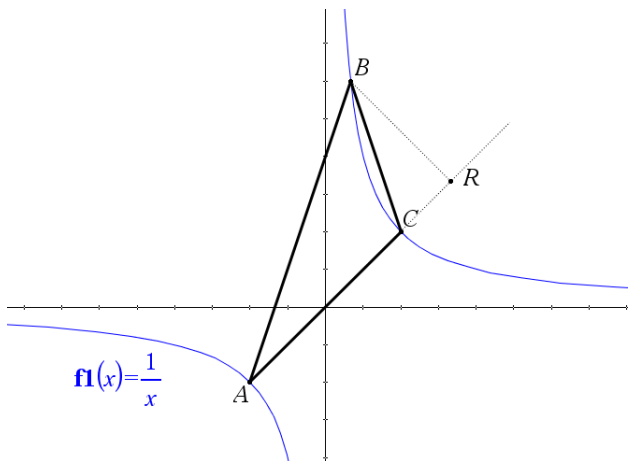
Point Q is such that AQ is perpendicular to QC.

Note: QC is another altitude of △ABC.

$f1(x) = \frac{1}{x}$

Question 13

Determine the value of *m* such that *m*AQ·CQ = 0 and hence define vector CQ

A = (-1, -1)

B = $\left(\frac{1}{3}, 3\right)$

C = (1, 1)

Point R is such that AR is perpendicular to RB.

Note: BR is the third altitude of △ABC.

$f1(x) = \frac{1}{x}$

Question 14

Determine the value of *s* such that *s*AC·BR = 0 and hence define the vector BR.

Question 15

Consider the values of *m*, *n*, and *s* established in the previous questions and their respective locations. By considering the values only, how is it possible to tell if the altitude (perpendicular) sits on the vector?

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Extension: Points of Intersection Investigation

It is possible to determine where each of the altitudes intersects one another. The diagram below shows the point of intersection between altitudes BR and AP at point S.
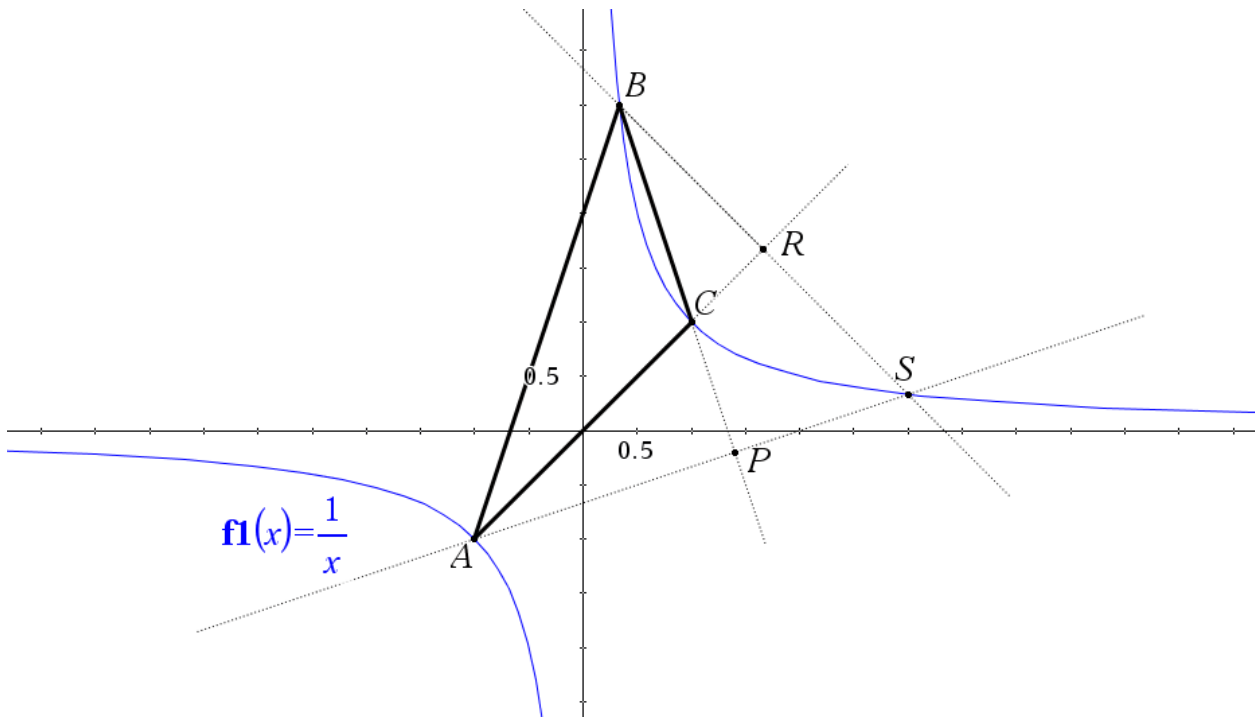
Important knowledge:

- BS is a scalar multiple of BR.
- AS is a scalar multiple of AP.
- OS = OB + BS and OS = OA + AS.

The above information can be used to determine the location of OS. Determine the location of S and explore the location of the point(s) of intersection for the altitude passing through C with those passing through A and B.

Vary the locations of points A, B and C and summarise the location of point S and the general vector defining OS.

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