Solving with Matrices

Teachers Teaching with Technology™
Professional Development from Texas Instruments

TI-Nspire   Investigation   Student   90 min

Introduction

Simultaneous linear equations can be solved by using algebraic approaches such as the elimination method or substitution method. They can also be solved graphically by finding points of intersection. Using CAS technology, a solve command can be used.

Another method of solving simultaneous linear equations requires the use of matrices. This approach involves representing the equations as rows of numbers and then manipulating these to reach solutions.

Part 1: Using matrices to solve a system of equations

Consider the simultaneous equations

$$5x - 3y = 22$$
$$x + 4y = -14$$

Each equation can be represented by a row of numbers with the coefficients of x coming first, followed by the coefficients of y and finally the constant terms. These rows make up a matrix as shown.

$$\begin{bmatrix} 5 & -3 & 22 \\ 1 & 4 & -14 \end{bmatrix}$$

In this matrix, the first row represents the first equation and the second row represents the second equation. Although this is the usual convention, it is equally correct to represent these simultaneous equations by the matrix

$$\begin{bmatrix} 1 & 4 & -14 \\ 5 & -3 & 22 \end{bmatrix}$$

Question 1.

Each pair of simultaneous equations below can be represented by a matrix. In each case, write down the matrix that corresponds to the pair of equations.

a) $3x + 5y = 21$
      $6x - 2y = 6$

b) $6x + 5y = 1$
      $x - 7y = 8$

$$\begin{bmatrix} 3 & 5 & 21 \\ 6 & -2 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 5 & 1 \\ 1 & -7 & 8 \end{bmatrix}$$

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Once a matrix representing the simultaneous equations has been created it is possible to transform this matrix to obtain the solutions for x and y. To do this, the **rref** command is used.

It has already been shown that simultaneous equations $5x - 3y = 22$ and $x + 4y = -14$ can be represented

by the matrix $\begin{bmatrix} 5 & -3 & 22 \\ 1 & 4 & -14 \end{bmatrix}$

The screenshot shows how the **rref** command is used to transform the matrix.
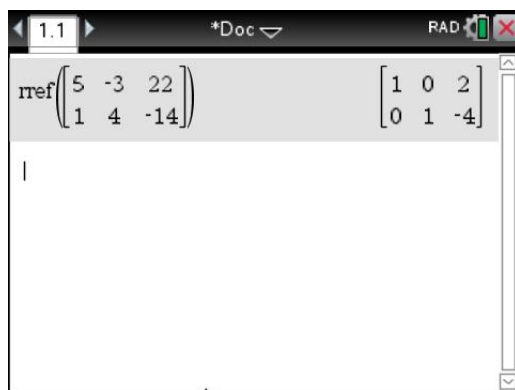
The transformed matrix represents the equations
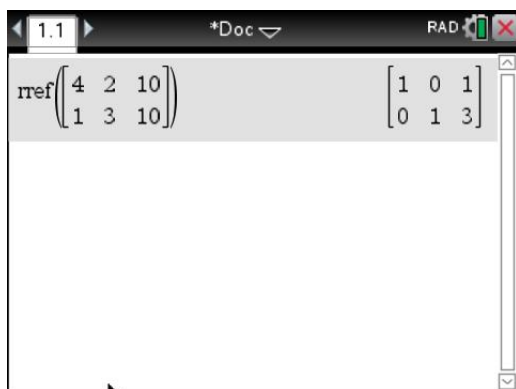
$$1x + 0y = 2$$
$$0x + 1y = -4$$

Therefore $x = 2$ and $y = -4$ is the simultaneous solution to these equations.
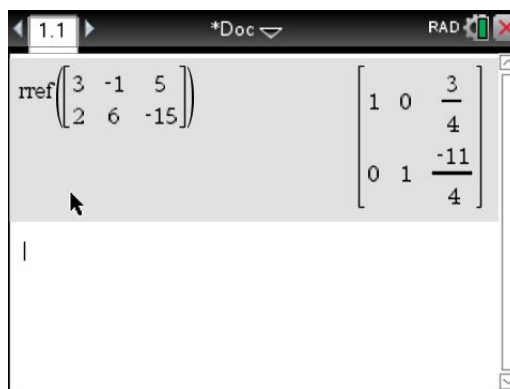
This solution can be checked using substitution.

- $5x - 3y = 5 \times 2 - 3 \times -4 = 10 + 12 = 22$, so $5x - 3y = 22$
- $x + 4y = 2 + 4 \times -4 = 2 - 16 = -14$, so $x + 4y = -14$

Question 2.

In each of the following, the **rref** command has been used to solve simultaneous equations. In each case, write down the equations being solved and their simultaneous solution.

Equations: $4x + 2y = 10$ and $x + 3y = 10$

Solution: $x = 1$ and $y = 3$

Equations: $3x - y = 5$ and $2x + 6y = -15$

Solution: $x = 3/4$ and $y = -11/4$

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Part 2: Using the 'rref' command

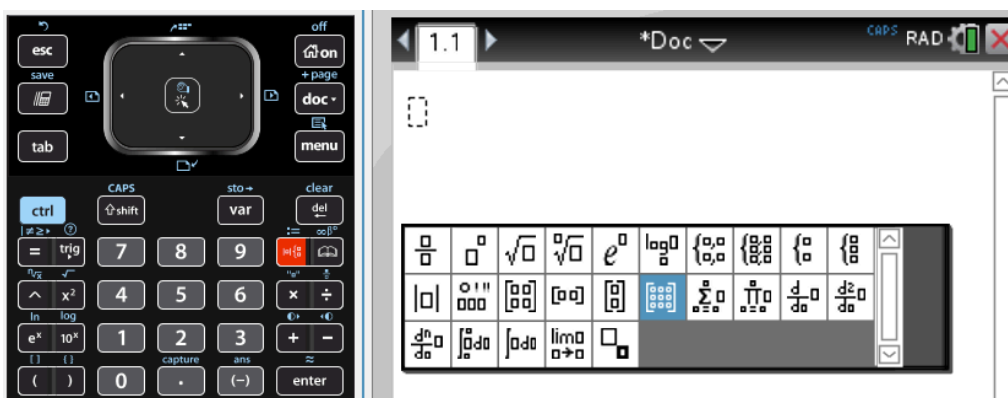Now consider the following simultaneous equations.

$$x - 2y = 10$$
$$2x + 4y = 16$$

Create a matrix with rows that are the coefficients of *x* and *y* and the constant terms in the equations.

$$\begin{bmatrix} 1 & -2 & 10 \\ 2 & 4 & 16 \end{bmatrix}$$

To enter this matrix on the TI-Nspire CAS:

- Press **HOME-1** to create a new document, and then press **1** to add a **Calculator** page.

- Press the **Templates** key (see screen above).

- Select the '**Create a Matrix**' icon
  (it looks like a 3 by 3 matrix – see screen above)

- In the dialog box that follows

  – For the number of rows, type 2.

  – For the number of rows, type 3.

- Press **ENTER** to create the 2 by 3 matrix template

- Type in the required values, then press **ENTER**.
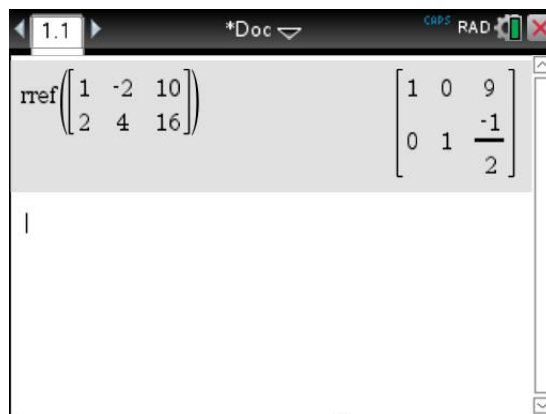
To apply the rref command, type $\text{rref}\left(\begin{bmatrix} 1 & -2 & 10 \\ 2 & 4 & 16 \end{bmatrix} \right)$

[Note: The **rref** command can also be accessed via the Catalog, or just by typing in the letters directly.]

The resultant screen shows that the **rref** command has transformed the original pair of equations into

$$1\,x + 0\,y = 9$$
$$0\,x + 1\,y = -1/2$$

So the solution is x = *9* and *y* = − 1/2.

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Question 3.

Use the **rref** command to solve each set of simultaneous equations below.

a) $3x + 5y = 21$ and $6x − 2y = 6$ *x = 2, y =3*

b) $6x + 5y = 1$ and $x − 7y = 8$ *x = 1, y = −1*

c) $x + 3y = −2$ and $2x − 5y = 18$ *x = 4, y = −2*

d) $4x + 3y = 3$ and $2x − 6y = −11$ *x = 1/2, y = 1/3*

The same method can be used to solve larger systems of linear equations.

Question 4.

Use the **rref** command to find values simultaneous equations of the pronumerals that satisfy each of the following systems of equations

a) $3x + 2y − z = 5$ *x = 1, y = 6/5, z = 2/5*
     $4x + 3y + z = 8$
     $2x + 2y − z = 4$

b) $2x + 3y + 2z = 10$ *x = 42, y = 34, z = −88*
     $−x + 4y + z = 6$
     $4x − 2y + z = 12$

Question 5.

What relationship exists between the number of equations and the number of unknowns?
The number of equations is equal to the number of unknowns.

Part 3: How many solutions are possible?

So far, all of the systems of equations have a single solution. But this is not true for all such systems. In this part of the exploration, we look at the different types of solutions, and how the **rref** command handles each situation. To begin, consider the following system of equations.

$$x + y = 1$$
$$x − y = 3$$

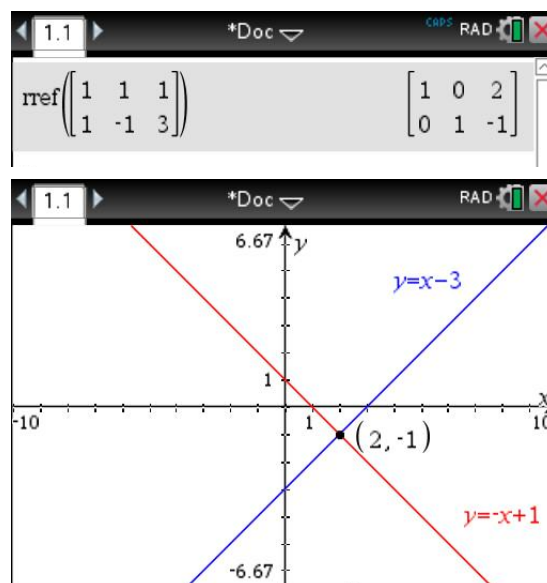Using the **rref** command, this system has a solution at $x = 2$ and $y = −1$ (see screen).

Graphically, we can represent this solution as the point at which the two associated lines intersect.

If we rewrite the two equations in the form y = mx + c, we get the following.

$x + y = 1 \Rightarrow y = −x + 1$

$x − y = 3 \Rightarrow y = x − 3$

If we graph these two equations, we would find that they intersect at the point (2, −1) as the screen shows. This is an example of a system of equations that has a **unique solution**.

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Now consider another system of equations.

$$x + y = 1$$
$$x + y = 3$$

Using the **rref** command, the calculator reports that this system has a 'solution' at $x + y = 0$ and $0 = 1$

As the equation $0 = 1$ is false, we can say that this system has **no solution**.

Graphically, the solution to a pair of linear simultaneous equations is the point at which the two associated lines intersect. If we rewrite the two equations in the form $y = mx + c$, we get the following.

$x + y = 1 \Rightarrow y = -x + 1$ and

$x + y = 3 \Rightarrow y = -x + 3$

By graphing these two equations, it can be observed that the lines are parallel (the gradients are both –1), so they never intersect! This is an example of a system of equations that has no solution.
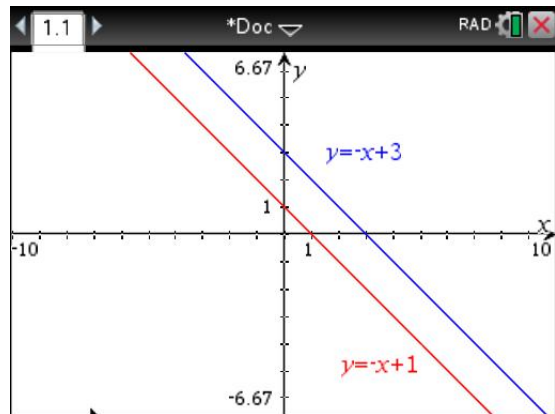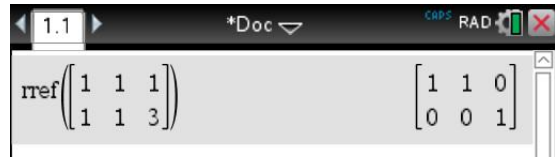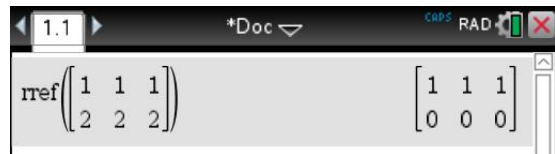
Finally consider the following system of equations.

$$x + y = 1$$
$$2x + 2y = 2$$

Using the **rref** command, this system has a solution at $x + y = 1$ and $0 = 0$

The statement $0 = 0$ is true and there exists an infinite set of solutions that satisfy the equation $x + y = 1$, some of which include, $x = 0$ and $y = 1$, $x = 1$ and $y = 0$, $x = -2$ and $y = 3$, $x = 0.5$ and $y = 0.5$. This is why the **rref** command has been unable to reduce the system to a unique solution. Instead it is reporting that there is more than one solution, in fact any pair of x and y values that add to 1!

Graphically, we can represent this solution as any point at which the two associated lines intersect. That is, if we rewrite the two equations in the form $y = mx + c$, we get the following.

$x + y = 1 \Rightarrow y = -x + 1$

$2x + 2y = 2 \Rightarrow 2y = -2x + 2 \Rightarrow y = 1/2(-2x + 2)$

If we graph these two equations, we would find that the graphs are identical (they have the same graph), as the screen shows. This is an example of a system of equations that has infinite solutions.

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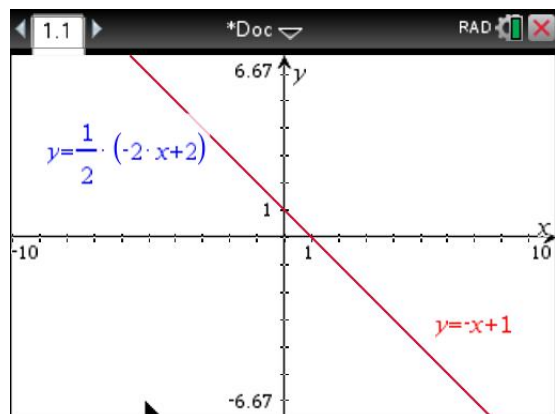
Question 6.

For the final case (infinite solutions), explain how the second equation is related to the first equation.
The second equation is the same equation as the first equation, just in another form. That is, it is an equivalent equation, made different by each term in the second equation being double the corresponding term in the first equation.

Question 7.

For the following systems of equations, use the **rref** command to determine whether they have a unique solution, no solutions, or infinite solutions. If there is a unique solution, give this solution. If there are infinite solutions, give the equation that describes all the values of x and y that would provide solutions.

a) $x + y = 1$ Unique solution; $x = 0$, $y = 1$
    $-x + 2y = 3$

b) $x + y = 1$ No solution
    $-x - y = 2$

c) $2x + y = 1$ Unique solution; $x = -1/3$; $y = 5/3$
    $x + 2y = 3$

d) $2x + 2y = 2$ Infinite solutions; $x + y = 1$
    $3x + 3y = 3$

Challenge

Determine the values of m and n for which the system

$$3x + 3y = 6$$
$$mx + y = n$$

will have

a) a unique solution   Unique solution if $m \neq 1$

b) no solutions   No solution if $m = 1$ and $n \neq 2$

c) infinite solutions.   Infinite solutions if $m = 1$ and $n = 2$

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Teacher notes

- This task is suitable for students in Year 11 mathematics, who have completed some work on solving pairs of simultaneous linear equations. It can be completed in a double lesson.

- Students will need to become familiar with entering a matrix of order m by n into the calculator, but thereafter very few new procedures are required.

- No prior experience of matrices is assumed or is needed. The matrices are used as a relevant representational form for the algorithm.

- The focus is on the use and interpretation of a matrix solving command **rref**, short for reduced row echelon form. The **rref** command employs an iterative algorithm that reduces a system of linear equations to equivalent and unique equations in their simplest form.

- As is discovered in Part 3, this **rref** algorithm is also capable of revealing any inconsistency or lack of uniqueness in the solutions. It provides some meaning to solution sets by providing a solution 'equation' for cases of infinite solutions, and an inconsistent (or false) equation for cases with no solution.

- Note that using the **rref** command also eliminates the problem of error messages such as 'Singular Matrix', which will occur when directly using matrix inverses to assist in the solving process.

- If the treatment of non-unique solutions is not required, only Parts 1 and 2 need be completed.

- In Part 3, there are references to graphical representations of the solution set; however, it is not expected that students attempt any graphical work on the calculator. It is included to help students to visualise the meaning of unique, none and infinite solutions.

- The challenge is only for the mathematically able students, but it leads naturally from Part 3 of the exploration. The effect of parameters on the nature of the solution is explored in greater depth in some Year 11–12 mathematics subjects.

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