Factors that Count

Teacher Notes

Texas Instruments
Your Passion. Our Technology. Student Success.™

7 **8** 9 10 11 12

TI-Nspire CAS | Investigation | Student | 120min

Aim

The number 12 has six factors: 1, 2, 3, 4, 6 and 12. The number 36 has more factors. Which number would have the greatest number of factors: 36 or 5929? This investigation explores how to determine the quantity of factors for any given number without listing each individual factor.

Number & Algebra - Year 7: Number and Place Value

Investigate index notation and represent whole numbers as products of powers of prime numbers (ACMNA149)

- solving problems involving lowest common multiples and greatest common divisors (highest common factors) for pairs of whole numbers by comparing their prime factorisation
- applying knowledge of factors to strategies for expressing whole numbers as products of powers of prime factors, such as repeated division by prime factors or creating factor trees

ScOT: Indices

Equipment

For this activity you will need:

- Playing cards / Numbered counters
- TI-Nspire CAS
- TI-Nspire CAS file (tns): Factors that Count

Preliminary Investigation:

1. Construct factor trees for each of the following numbers:

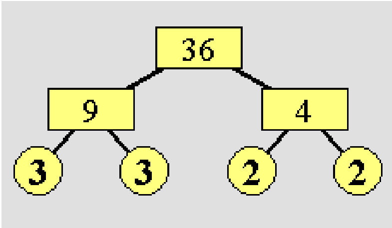
   a) 36 b) 100 c) 196 d) 441

Answer(s) & Comment:

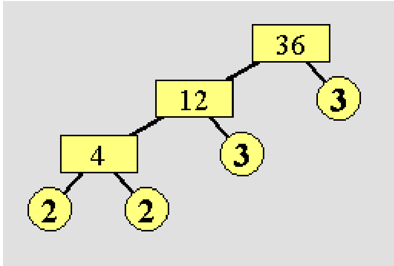
The factor trees will vary depending on the original factors selected, however the prime factorisation at the bottom of the tree will be the same.

Samples factor trees for 36:

Author: P Fox



Initial factors of 9 and 4 produces a tree with a different appearance than 12 and 3, however the prime factorisation remains the same. (Fundamental Theorem of Arithmetic)



Unlike the image shown, encourage students to extend 'redundant' branches so the factor product at any level on the tree produces the original quantity: 3 x 3 x 2 x 2 = 36.

Comments:

There are a number of websites that provide an interactive space where students can populate a tree diagram. ie: http://nlvm.usu.edu/en/nav/frames_asid_202_g_2_t_1.html?from=category_g_2_t_1.html

2. Write down the prime factorisation for each of the numbers 36, 100, 196 and 441.

Answer(s):

Students should be encouraged to write the prime factorisation as:

36	$= 2 \times 2 \times 3 \times 3$	100	$= 2 \times 2 \times 5 \times 5$	196	$= 2 \times 2 \times 7 \times 7$	441	$= 2 \times 2 \times 11 \times 11$
	$= 2^2 \times 3^2$		$= 2^2 \times 5^2$		$= 2^2 \times 7^2$		$= 2^2 \times 11^2$

This helps reinforce index notation as well as preparing for later questions that focus on the index representation.

3. Using a pack of playing cards (or numbered counters), determine all the factors for each of the numbers 36, 100, 196 and 441. Record all arrangements of the cards used to determine the factors for each number.
Remember to include one and the original number in the factor count.

Comments:

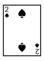
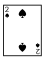
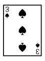
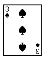
Using the playing cards is NOT a trivial exercise! It is the physical manipulation of the playing cards that encourages the systematic manipulation of the factors and helps students identify that it is not the base number that influences the number of factors, rather the frequency of its occurrence, the index. The manipulation of 2, 2, and 3 will produce the same number of combinations as 3, 3 and 5, hence the number of factors for $2^2 \times 3$ will be the same as for $3^2 \times 5$.

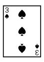
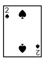
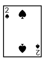
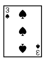
Author: P Fox

Students should be encouraged to write the prime factorisation including appropriate set notation:

Factors		Arrangements	
36	{1, 2, 3, 4, 6, 9, 12,18,36}	2 x (2 x 3 x 3) 3 x (2 x 2 x 3) (2 x 2) x (3 x 3) (2 x 3) x (2 x 3)	2 x 18 3 x 12 4 x 9 6 x 6
100	{1, 2, 4, 5, 10, 20, 25, 50, 100}	2 x (2 x 5 x 5) (2 x 2) x (5 x 5) 5 x (2 x 2 x 5) (2 x 5) x (2 x 5)	2 x 50 4 x 25 5 x 20 10 x 10
196	{1, 2, 4, 7, 14, 28, 49, 98}	2 x (2 x 7 x 7) (2 x 2) x (7 x 7) 7 x (2 x 2 x 7) (2 x 7) x (2 x 7)	2 x 98 4 x 49 7 x 28 14 x 14
441	{1, 3, 7, 9, 21, 49, 63, 147, 441}	3 x (3 x 7 x 7) 7 x (3 x 3 x 7) (3 x 3) x (7 x 7) (3 x 7) x (3 x 7)	3 x 147 7 x 63 9 x 49 21 x 21

Examples: Factors of 36

		×		×		×	
	2	×	(2	×	3	×	3)
Factor Pair:	2	×	18				

		×		×		×	
	3	×	(2	×	2	×	3)
Factor Pair:	3	×	12				

Author: P Fox

4. What do you notice about the quantity of factors for the numbers 36, 100, 196 and 441?

Comments:

The quantity of factors is the same for all of these numbers. It is worth questioning students "what is the same about all of the prime factorisations" and "what is different" to alert them to the similarity of the indices and differences in the bases.

Investigation – The factor Sleuth:

5. Use questions 1 to 4 as a guide to investigate the following numbers: 24, 54, 250 and 1029. Determine the prime factorisation and the quantity of factors for each of these numbers.

Comments:

Students should draw factor trees for each of these and include the prime factorisation as per the previous questions. Students should start to identify similarities in these representations:

$24 = 2 \times 2 \times 2 \times 3 = 2^3 \times 3$

$54 = 2 \times 2 \times 2 \times 7 = 2^3 \times 7$

$250 = 2 \times 5 \times 5 \times 5 = 2 \times 5^3$

$1029 = 3 \times 7 \times 7 \times 7 = 3 \times 7^3$

Each of these numbers has 8 factors, note the same indices for each prime factorisation.

6. Use questions 1 to 4 as a guide to investigate the following numbers: 80, 162, 405 and 1250. Determine the prime factorisation and the quantity of factors for each of these numbers.

Comments:

Students should draw factor trees for each of these and include the prime factorisation as per the previous questions.

$80 = 2 \times 2 \times 2 \times 2 \times 5 = 2^4 \times 5$

$162 = 2 \times 3 \times 3 \times 3 \times 3 = 2 \times 3^4$

$405 = 3 \times 3 \times 3 \times 3 \times 5 = 3^4 \times 5$

$1250 = 2 \times 5 \times 5 \times 5 \times 5 = 2 \times 5^4$

Each of these numbers has 10 factors, note the same pair of indices for each prime factorisation.

The numbers from Question 1: 36, 100, 196 and 441 all have something in common based on their prime factorisation, let this collection of numbers be identified as 'Group 1'. The numbers from Question 5: 24, 54, 250 and 1029 also have something in common, let this collection of numbers be identified as 'Group 2'. The numbers from Question 6: 80, 162, 405 and 1250 belong to 'Group 3'.

7. Identify which groups each of the following numbers belong to and determine the quantity of factors for each: 88, 104, 875, 484, 1375, 3773 and 3025.

Author: P Fox

Comments:

Students should draw factor trees for each of these and include the prime factorisation as per the previous questions.

$88 = 2 \times 2 \times 2 \times 11 = 2^3 \times 11$ (Group 2 – indices are 3 and 1 therefore 8 factors)

$104 = 2 \times 2 \times 2 \times 13 = 2^3 \times 13$ (Group 2 – indices are 3 and 1 therefore 8 factors)

$484 = 2 \times 2 \times 11 \times 11 = 2^2 \times 11^2$ (Group 1 – indices are 2 and 2 therefore 9 factors)

$1375 = 5 \times 5 \times 5 \times 11 = 5^3 \times 11$ (Group 2 – indices are 3 and 1 therefore 8 factors)

$3773 = 7 \times 7 \times 7 \times 11 = 7^3 \times 11$ (Group 2 – indices are 3 and 1 therefore 8 factors)

$3025 = 5 \times 5 \times 11 \times 11 = 4^2 \times 11$ (Group 2 – indices are 3 and 1 therefore 8 factors)

8. Determine three numbers that have exactly nine factors and explain how you found these numbers.
   You can not use numbers that have already been included in this activity.

Comments:

Students may not yet have articulated or created a formal rule, they may simply refer to previous Prime Factorisations that had 9 factors, such as: $2^2 \times 3^2 = 36$ or $2^2 \times 5^2 = 100$. Students should use other bases such as: $17^2 \times 13^2$ but NOT $4^2 \times 5^2$, since 4 is not prime.

The answer will be in the form: $a^2 \times b^2$ where *a* and *b* are prime.

It is also noteworthy that all the answers will be perfect squares.

9. Determine three numbers that have exactly eight factors and explain how you found these numbers.
   You can not use numbers that have already been included in this activity.

Comments:

Once again, students may not yet have articulated or created a formal rule, they may simply refer to previous Prime Factorisations that had 8 factors, such as: $2 \times 3^3 = 54$ or $2^3 \times 5 = 40$. Students should use other bases such as: 17×13^3 but NOT $4^3 \times 5$ or 4×5^3, since 4 is not prime.

The answer will be in the form: $a^3 \times b$ where *a* and *b* are prime.

10. Explain how the prime factorisation of a number helps identify the quantity of factors.

Comments:

The numbers: $2 \times 3^3 = 54$ and $2^3 \times 5 = 40$ both have 8 factors. The bases are different but the quantity of factors are the same. The numbers: $2 \times 3^3 = 54$ and $2^4 \times 3 = 48$ have the same bases but a different quantity of factors. So the base numbers seem to have no bearing on the number of factors. The indices control the quantity of factors. Students may not have articulated or formulated a rule, however at this point they should be able to identify that it is the indices that influence the number of factors. The rearranging of playing cards is designed to get students to that understanding.

Author: P Fox

Investigation – Factors Rule:

It is time to formalise a rule that identifies the quantity of factors for any given number. The rule uses the prime factorisation of the number.

11. Investigate each of the following prime factorisations and determine the quantity of factors for each.

The first one has been done for you.

Number	Prime Factorisation	Factors	Qty of Factors	Indices	Bases
108	$2^2 \times 3^3$	{1,2,3,4,6,9,12,18,27,36,54,108}	12	2, 3	2, 3
675	$5^2 \times 3^3$	{1, 3, 5, 9, 15, 25, 27, 45, 75, 135, 225, 675}	12	2, 3	3, 5
1125	$5^3 \times 3^2$	{1, 3, 5, 9, 15, 25, 45, 75, 125, 225, 375, 1125}	12	3, 2	3, 5
392	$7^2 \times 2^3$	{1, 2, 4, 7, 8, 14, 28, 49, 56, 98, 196, 392}	12	2, 3	2, 7
968	$11^2 \times 2^3$	{1, 2, 4, 8, 11, 22, 44, 88, 121, 242, 484, 968}	12	2, 3	2, 11
726	$2 \times 3 \times 11^2$	{1, 2, 3, 6, 11, 22, 33, 66, 121, 242, 363, 726}	12	1, 1, 2	2, 3, 11
132	$2^2 \times 3 \times 11$	{1, 2, 3, 4, 6, 11, 12, 22, 33, 44, 66, 132}	12	2, 1, 1	2, 3, 11
198	$2 \times 3^2 \times 11$	{1, 2, 3, 6, 9, 11, 18, 22, 33, 66, 99, 198}	12	1, 2, 1	2, 3, 11
396	$2^2 \times 3^2 \times 11$	{1, 2, 3, 4, 6, 9, 11, 12, 18, 22, 33, 36, 44, 66, 99, 132, 198, 396}	18	1, 2, 2	2,3,11
1100	$2^2 \times 5^2 \times 11$	{1, 2, 4, 5, 10, 11, 20, 22, 25, 44, 55, 100, 110, 220, 275, 550, 1100}	18	2, 2, 1	2, 5, 11
700	$2^2 \times 5^2 \times 7$	{1, 2, 4, 5, 7, 10, 14, 20, 25, 28, 35, 50, 70, 100, 140, 175, 350, 700}	18	2, 2, 1	2, 5, 7

Comments:

The purpose of the table is to have students focus on the indices and number of factors. The table helps isolate the indices and the quantity of factors. It is aimed at helping students reduce the 'noise' of irrelevant information. How can 9 be produced when the indices are 2 and 2. Can we use a similar formulation to create 12 from 2 and 3 ?

12. Study carefully the indices and the number of factors. Write down a 'conjecture' (proposed rule) that relates the indices to the number of factors.

Comments:

This question requires students to articulate either literally or symbolically the rule for determine the number of factors.

ie: "Add one to each of the indices and then calculate the product..."

ie: "If the prime factorisation is in the form: a^n x b^m then the number of factors can be calculated using:

(m + 1) x (n + 1)

Author: P Fox

13. Generate your own table of numbers to explore. Explain your selection of numbers and how they were generated.

Comments:

Students should consider numbers with more than two prime factors and also a single prime factor.

14. Use your table of numbers to check or validate your rule from question 12.
Note: If your rule from question 12 does not work, you will need to re-write it.

Comments:

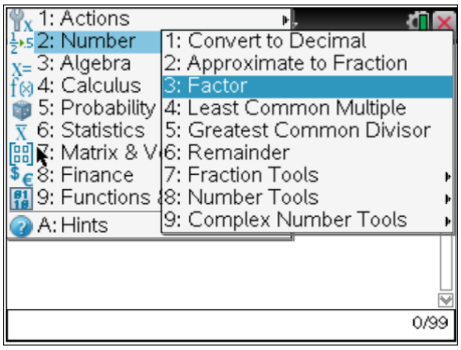
Mathematically, students are substituting numbers into a formula. The validation of the rule should consider a range of carefully selected examples, such as more than two prime factors, situations where all the indices equal to one. Students that complete a table of values with the same index combination are not 'exhaustively' testing their rule or considering possible situations where the rule won't work. This question however does not provide specific instruction like earlier questions; students often have difficulties with the open-ended nature of the question.

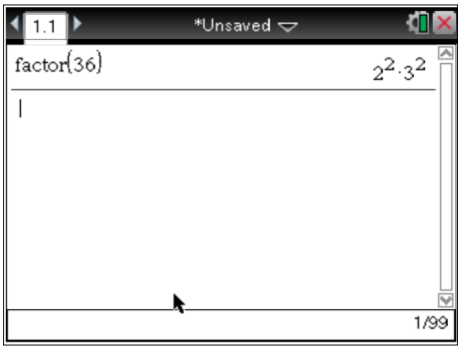
Technology:

TI-Nspire CAS can be used to explore the prime factorisation of a number:

factor(36)

From the **Number** (or **Algebra**) menu, select **Factor**.





Author: P Fox