Concepts

The Fundamental Theorem of Calculus, Part 1, provides the connection between differential and integral calculus. If f is a continuous function on $[a,b]$, then the function g defined by

$$g(x) = \int_a^x f(t)dt \qquad a \le x \le b$$

is continuous on $[a,b]$ and differentiable on (a,b), and $g'(x) = f(x)$. In words, this theorem says that the derivative of a definite integral with respect to its upper limit is the integrand evaluated at that upper limit.

Here is some other common notation to illustrate the conclusion of the FTC, Part 1.

$$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$$

An interpretation of this expression is that integration and differentiation are inverse operations; what one does, the other undoes.

Course and Exam Description Unit

Section 6.4: The Fundamental Theorem of Calculus and Definite Integrals

Calculator Files

FundamentalTheorem.tns

Using the Document

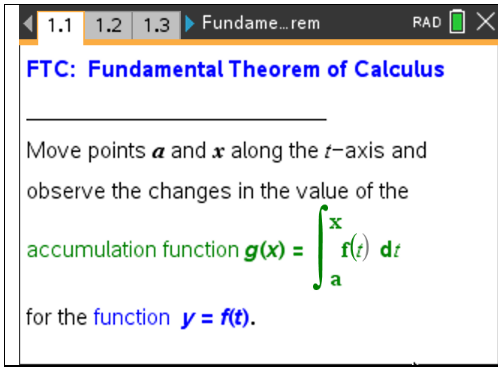
FundamentalTheorem.tns: On page 1.2, the function f is defined in a Math Box. The default definition for f is $f(t)=\dfrac{6t^2-9t-15}{40}$. This expression can be changed by the user to allow for more in-depth discussions and conceptual questions concerning the Fundamental Theorem of Calculus.

The graph of f is displayed on Page 1.3. The values a and x can be changed by grabbing the corresponding point and dragging along the horizontal axis. The value $g(x)$ is displayed in the bottom pane.

Page 1.4 shows a graph of f and a graph of g, aligned horizontally. The values a and x can be changed on the graph of f by grabbing the corresponding point and dragging along the horizontal axis. Page 1.5 shows the graph of g in the top pane, and the graph of its derivative in the bottom pane.

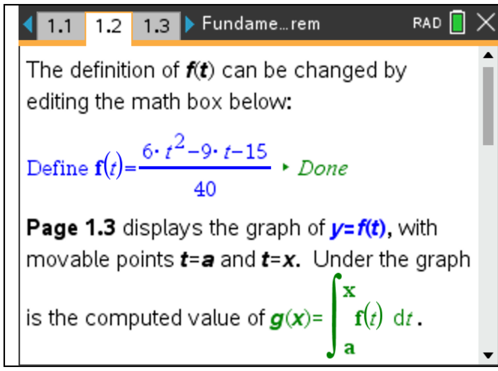
Problem 2 presents a special application in which the accumulation function is used to define the natural logarithm function. Pages 2.3 and 2.4 present graphical evidence to confirm the definition of the natural logarithm function and its derivative.

Page 1.1



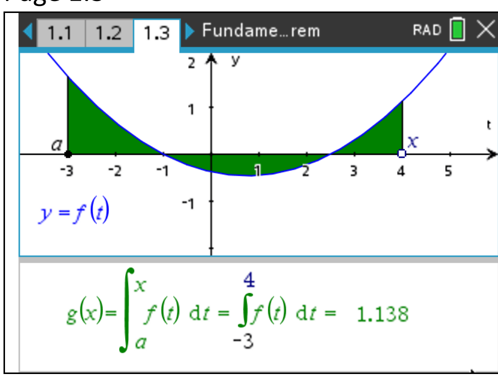
In Problem 1, the accumulation function g is examined graphically for a quadratic function f. The user can grab and move the points on the graph representing a and x.

Page 1.2



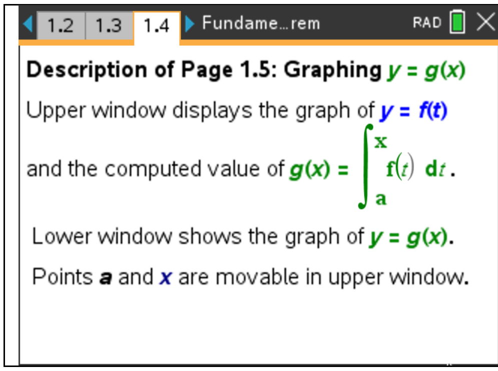
The function f is defined in a Math Box. The default definition is shown in the figure to the left. The characteristics of this function provide an opportunity to discuss and discover the relationship between f and g. However, the user can change this function to allow for other questions. There is also a brief description of the graphs on Page 1.3 here.

Page 1.3



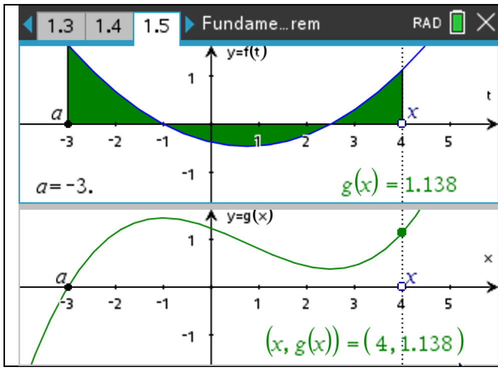
The graph of the function f is shown in the top pane. The values a and x can be changed on the graph of f by grabbing the corresponding point and dragging along the horizontal axis. The shaded region in the graph is a geometric representation of the value of $g(x)$. The value of the accumulation function, $g(x)$, for the current values of a and x is shown in the bottom pane.

Page 1.4



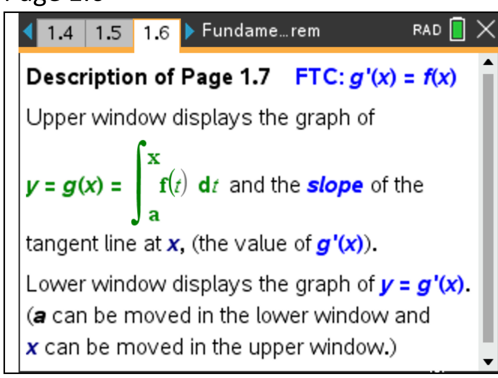
This page presents a brief description of the graphs on Page 1.5. The top pane of Page 1.5 shows a graph of the function f, the geometric shaded region representing the value $g(x)$, and the numerical value $g(x)$. The bottom pane shows the graph of g. The values a and x can be changed in the top pane by grabbing the corresponding point and dragging along the horizontal axis.

Page 1.5



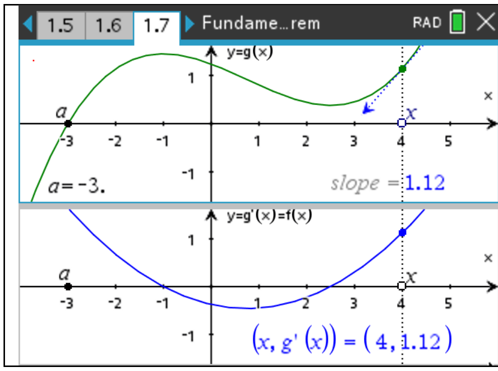
The graph of f is displayed in the top pane. The points representing a and x can be moved in the top pane only. The shaded region is a geometric interpretation of the value $g(x)$. The current value of g is shown in the top pane, and the corresponding point on the graph of g is shown in the bottom pane. The graph of g is displayed in the bottom pane; this graph updates dynamically as a or x changes. The graphs of f and g align horizontally so that the connections between the two functions are more apparent.

Page 1.6



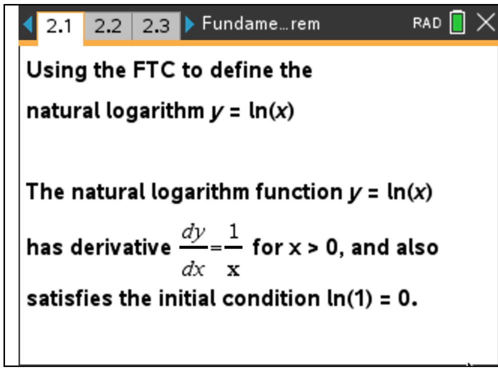
This page presents a brief description of the graphs on Page 1.7. The top pane of Page 1.7 shows a graph of the accumulation function g, the tangent line to the graph of g at x, and the slope of the tangent line. The bottom pane shows the graph of g'. The point representing a can be moved in the bottom pane and the point representing x can be moved in the top pane.

Page 1.7



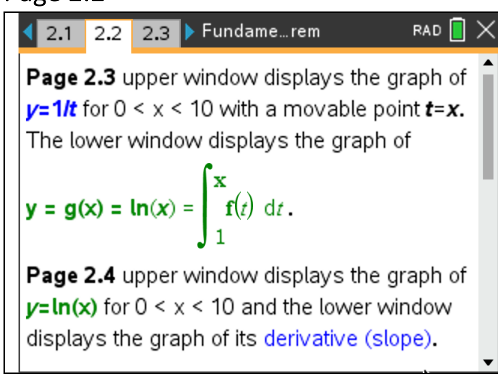
The graph of the accumulation function g, the tangent line to the graph of g at x, and the slope of the tangent line are displayed in the top pane. The value of a is displayed in the lower left, and the value of the slope is shown in the lower right. The point representing x can be moved in the top pane. The bottom pane shows a graph of the derivative g' and the coordinates of the point on the graph of g'. The point representing a can be moved in the bottom pane.

Page 2.1



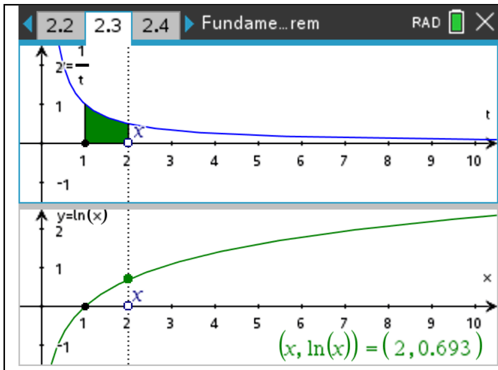
This page presents a general description of calculator Problem 2. Using the information discovered in Problem 1, we can define the natural logarithm function using an accumulation function.

Page 2.2



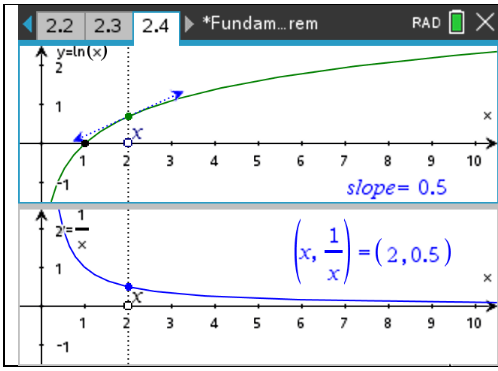
This page presents a brief description of the graphs displayed on Pages 2.3 and 2.4. These pages are similar to Pages 1.5 and 1.7, respectively, but use the function $f(t) = \frac{1}{t}$, and appropriate graphing windows.

Page 2.3



The graph of $y = \frac{1}{t}$ is displayed in the top pane. The value of a is fixed at $a = 1$. The point representing x can be moved in the top pane only. The shaded region is a geometric interpretation of the value $g(x) = \ln x$. The corresponding point on the graph of g is shown in the bottom pane. The graph of g is displayed in the bottom pane; this pane updates dynamically as x changes. The graphs of $y = \frac{1}{t}$ and $y = \ln x$ align horizontally so that the connections between the two functions are more apparent.

Page 2.4



The graph of the accumulation function, in this case, $y = \ln x$, the tangent line to the graph of $y = \ln x$ at x, and the slope of the tangent line are displayed in the top pane. The value of the slope is shown in the lower right. The point representing x can be moved in the top pane only. The bottom pane shows a graph of the derivative g', or $y = \frac{1}{x}$, and the coordinates of the point on the graph of g'. As x moves, the slope of the tangent line and the coordinates of the point are updated.

Suggested Applications and Extensions

Page 1.3

Use the default function f and the default value of a to answer Questions 1-5. Remember that g is a function of x (for a fixed value of a). The values of a and x can be manipulated, the value of $g(x)$ is displayed in the bottom pane, and the shaded region in the top pane represents the accumulated net area bounded by the graph of f and the horizontal axis from a to x.

1. Use geometry to estimate the value for $g(-1)$. Then move the point representing x to -1 on the horizontal axis to find the exact value. Is your estimate for $g(-1)$ an overestimate or underestimate? Why?
2. Use geometry to estimate the value for $g(2.5)$. Then move the point representing x to 2.5 on the horizontal axis to find the exact value. Explain why $g(2.5) < g(-1)$.
3. On what intervals is g increasing? Decreasing? What are the values of f on each of these intervals?

4. For what value(s) of x does g have a relative maximum? Relative minimum? Find $f(x)$ for each value of x at which g has a relative extrema. What does this suggest about the relationship between f and g?
5. Find the absolute maximum value and the absolute minimum value of g on the interval $[-3,5]$.
6. Let $a = 3$. Find $g(-1)$ and explain why $g(-1) > 0$ even though there is more shaded area below the horizontal axis than above.
7. Let $a = 3$. Find the absolute maximum value and absolute minimum value of g on the interval $[-3,5]$. How do these answers compare with those in Question 5?
8. Let $a = 0$ and find $g(-3)$. Let $a = -3$ and find $g(0)$. How do these two values compare? Is this result always true if the values of a and x are switched? Why or why not?

Page 1.5

Use the default function f and the default value of $a = -3$ to answer Questions 1-4. The bottom pane shows a graph of the function g.

1. Use the graph of g to find the intervals on which g is increasing. Decreasing. What are the values of f on each of these intervals? How do our answers compare with those in Question 3 above?
2. Estimate the slope of the tangent line to the graph of g and the value of f at $x = -1, 0, 3$. How do these values compare?
3. Find the intervals on which the graph of g is concave down. Concave up. Estimate the x-coordinate of the point inflection on the graph of g. What do you notice about the graph of f at this value? Explain the behavior of f around this value.
4. Find an equation of the tangent line to the graph of g at the point with x-coordinate 2.
5. Grab the point representing a in the top pane and move it slowly to the right, until $a = 5$. Describe how the graph of g changes as a increases from -3 to 5. Try to use the graph of f to explain how g changes.

Page 1.7

The graph of g is shown in the top pane, and the graph of g' is shown in the bottom pane.

1. Grab and move the point in the top pane representing x. Verify that the value of $g'(x)$ is the slope of the tangent line to the graph of g at x.
2. Compare the graph of g' in the bottom pane with the graph of f on page 1.5 in the top pane. What does this suggest about the relationship between f and g?
3. Grab and move the point in the bottom pane representing a. Explain why the graph of g changes but the graph of g' does not change.
4. Can you move the point representing a (in the bottom pane) such that the slope of the tangent line to the graph of g at $a = 0$ is -1? Why or why not?

Pages 2.3 and 2.4

The purpose of this example is to use the Fundamental Theorem of Calculus to define the natural logarithm function.

Page 2.3

1. Use the graph of $y = \frac{1}{t}$ and the definition of g to explain why the graph of $y = g(x) = \ln x$ is always increasing.
2. Use the graph of $y = \frac{1}{t}$ to give a geometric interpretation of the value of $\ln 4$ and $\ln(0.5)$.
3. Use the definition of the accumulation function to explain why $\ln 1 = 0$.
4. Use the graph of $y = \frac{1}{t}$ to describe the concavity of $y = \ln x$.
5. Find the values $\ln 2$ and $\ln(0.5)$. Explain how these two values are related in terms of accumulated area.

Page 2.4

1. Grab and move the point in the top pane representing x. Verify that the value of $g'(x) = \frac{1}{x}$ is the slope of the tangent line to the graph of $y = \ln x$ at x.
2. Use these graphs to confirm and state the relationship between the functions $y = \ln x$ and $y = \frac{1}{x}$.
3. Use these graphs and the definition of the accumulation function to explain why $\ln x < 0$ for $0 < x < 1$.
4. As x increases, describe the behavior of the slopes of the tangent lines to the graph of $y = \ln x$. Use the graph of $y = \frac{1}{x}$ to confirm this observation.
5. Find the slope of the tangent line to the graph of $y = \ln x$ at $x = 0.2$ and at $x = 5$. Explain how these values are related and why.