Amazing Chords

Teacher Notes and Answers

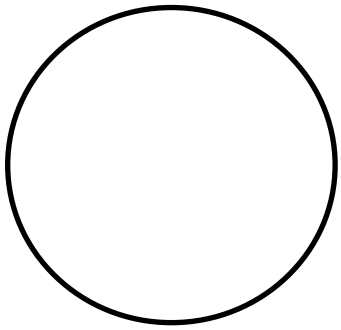
7 8 9 10 **11** 12

TI-Nspire Investigation Student 60 min

Introduction

Teacher Notes: This activity was inspired by Grant Sanderson (3 Blue 1 Brown) and the team at Numberphile. Their video on Betrand's paradox is excellent. This activity essentially moves students through the series of proofs completed by Grant in the video. The video can be shared with students once they have completed the activity.

Imagine a circle of radius one unit. Draw a chord (line connecting two points on the circumference of the circle) anywhere on the circle, then measure the length of the chord. Now imagine your friend also draws a chord somewhere else on the circle. Suppose you continue to draw chords on the circle, what would be the average length of these chords?



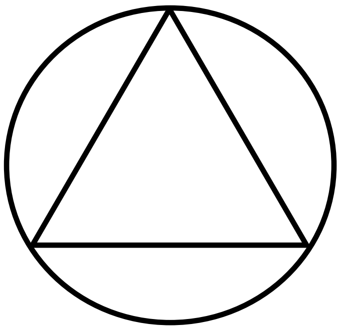
Question: 1.

Measure the diameter of the circle (opposite) and draw ten chords on the circle. Measure the length of each chord.

a) What is the average chord length?

Answers: Results will vary, particularly with the very low sample size and a *bias* towards longer chords.

Calculating the theoretical average chord length is somewhat challenging. The question can be made easier by adding some detail. An equilateral triangle has been drawn in the circle opposite. The aim is to compare the length of a randomly drawn chord to the side length of the equilateral triangle.



Question: 2.

What proportion of your randomly drawn chords are longer than the side length of the equilateral triangle? (Both circles are the same size)

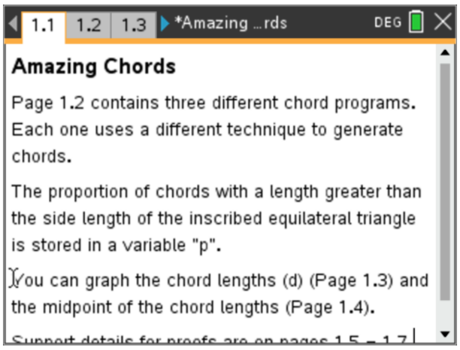
Answers: Results will vary. For a large sample, the result will be 1/3.
As noted in Question 1, when students draw the chords there is a bias towards longer chords as really short chords are harder to see and measure.

A small sample of size 10 is relatively small. To speed things up a couple of programs have been written into a TI-nspire™ to generate lots of chords. The following instructions provide a guide to work through the simulated and theoretical results to compute the proportion of random chords that are longer than the side lengths of the inscribed equilateral triangle.

Author: P. Fox

Chord - 1

Instructions:

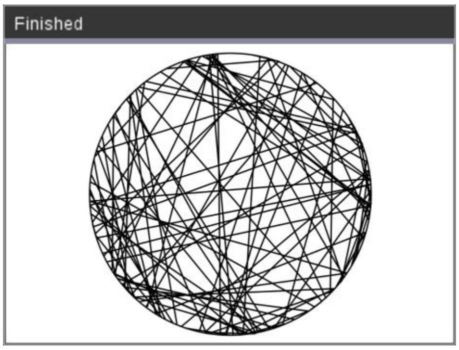


Open the TI-Nspire file: Amazing Chords.

Navigate to Page 1.2 and run the "Chord1" program.

To run the program press the [var] key and select Chord1. Press [enter] to run the program.

A prompt will appear requesting the number of chords to be generated. Start with 100 chords.



The screen shown opposite shows 100 randomly generated chords (program output). Press [esc] to return to the calculator application.

To see the proportion of chords that are longer than the side length of the inscribed equilateral triangle, press "P" followed by [enter].

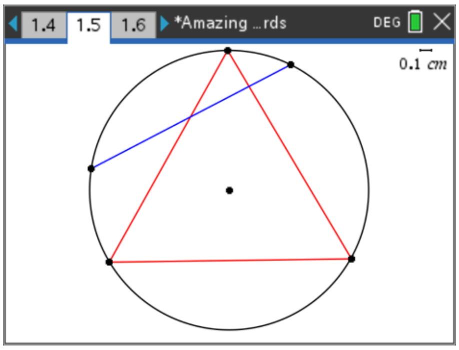
The length of each chord is stored in a list "d". The diameter of the circle in the simulation is 200 units. The distribution of the chord lengths can be seen on page 1.3.

Question: 3.

Run the program 10 times and record the proportion of chords that are longer than the side length of the equilateral triangle (p). Based on your results, what do you think the probability that a randomly generated chord will be longer than the side length of the inscribed equilateral triangle?

Answers: Results will vary. The mean should be approximately 1/3.

Teacher Notes: This is an opportunity to introduce sampling distributions. Run simulations of 100 and 400 and collect the results from students. There will be much less variation when students are generating samples of size 400 compared to samples of size 100. If TI-Navigator is available, consider doing a Poll (Lists), collect 10 results from each student, store as a table and generate dot-plots for both data sets: 100 and 400 simulations.



Navigate to page 1.5. A circle with radius 1 unit has been drawn. An equilateral triangle has been inscribed in the circle and a chord.

- Move the chord to wherever you want on the circle.
- Drag one of the triangle vertices so that it is on top of one end of the chord.
- Study this diagram carefully. Move the end of the chord that is not aligned to the triangle.
- Without measuring, how can you determine if your 'random' chord is longer or shorter than the side length of the triangle?

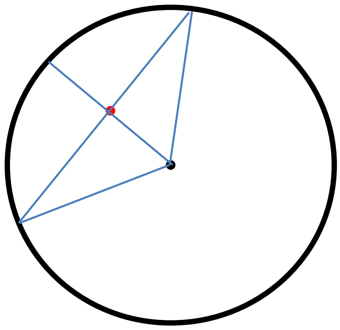
Question: 4.

Based on your experimentation (above), what proportion of randomly generated chord lengths will be longer than the side length of the inscribed equilateral triangle? [Justify your answer]

Answers: 1/3. Placing the point of the triangle on one end of the chord shows that the first and third arc (both equal in length) produce chord lengths smaller than the side length of the triangle. Chords terminating along the middle arc all have side lengths longer than the side length of the triangle. Assuming all points on the circumference are equally likely means 1/3 of chords produced this way will be longer than the side lengths of the triangle. This should align well with the data generated by the Chord1() program.

Author: P.Fox

Chord – 2



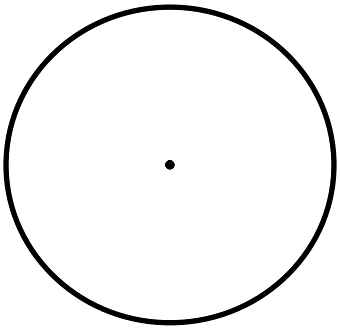
Another way to produce random chords is to place a point at some random location inside the circle and draw a chord through this point. For the purpose of this section, assume the random point represents the middle of the chord.

Question: 5.

Draw a diagram to help explain how the chord was drawn such that the randomly generated point was the midpoint of the chord.

Answers: Students should include the appropriate constructions. Draw a line from the centre of the circle through the random point. Construct the chord perpendicular to this line. Draw in the radii to create two congruent triangles. (Two corresponding sides and a right angle).

Question: 6.



Plot 5 points inside the circle and draw chords through each such that each random point is located at the midpoint of the corresponding chord.

a) Record the lengths of the chords.

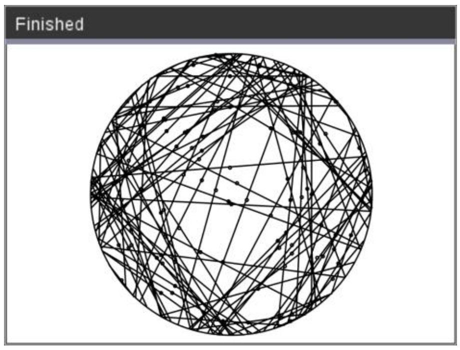
Answer: Answers will vary. Students should identify that points close to the centre of the circle produce longer chords, therefore the proportion of chords longer than the side length of the inscribed equilateral triangle will relate to the distance from the centre of the circle.

b) What proportion of your chords are longer than the side length of an inscribed equilateral triangle? (This circle is the same size as Question 1 & 2)

Answer: Again, answers will vary, with larger samples the result is approximately 25%.

c) Based on your sample, are both methods of generating chord lengths producing similar results?

Answer: Students should acknowledge that a significant variation will occur for the small sample size and be 'inconclusive' about the comparison.



Return to Page 1.2 and run the chord2() program. Generate 100 random chords. Press esc to return to the calculator application.

To see the proportion of chords that are longer than the side length of the inscribed equilateral triangle, press "P" followed by enter.

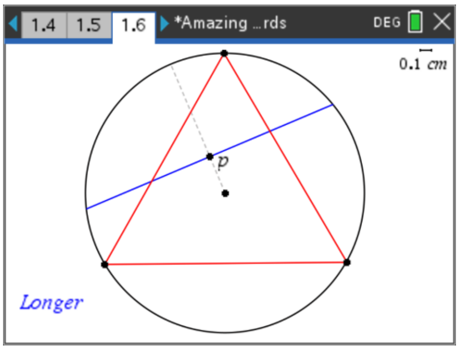
The length of each chord is stored in a list "d". The diameter of the circle in the simulation is 200 units, so as a percentage of the diameter compute: mean(d/200)

Question: 7.

Run the program 10 times generating 100 chords each time. Record the proportion of chords that are longer than the side length of the equilateral triangle (p). Based on your results, what do you think the probability that a randomly generated chord will be longer than the side length of the inscribed equilateral triangle using this approach?

Answer: The proportion is approximately 25%, significantly different than for the previous approach. Even when the standard error is accounted, students should be 'concerned' about this proportion!

Author: P.Fox



Navigate to page 1.6. Point P is the random point. The red triangle is an equilateral triangle inscribed in the circle. The length of the chord changes automatically as P is moved around.

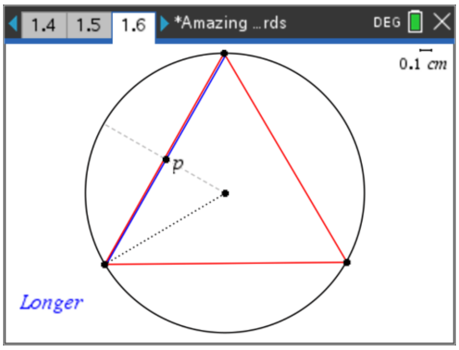
The text in bottom left corner of the screen indicates when the chord is *longer* than the side length and also *shorter*.

Move point P around the interior of the circle and establish an 'artificial' boundary whereby all chords generated through P will be longer than the side lengths of the triangle.

Question: 8.

Through what region can point P move to ensure all chord lengths generated in this region will be longer than the side lengths of the triangle?

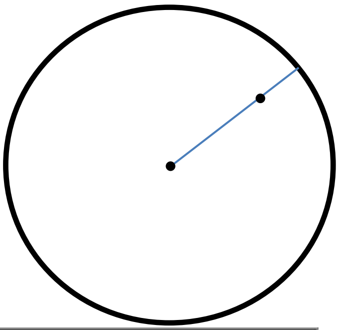
Answer: The point P can be moved within a circle of radius that is half the size of the circle's radius, this can be proven relatively easily.



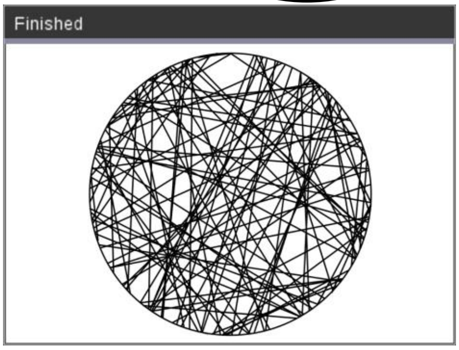
Place the point on the side of the equilateral triangle, the chord will run along the side of the triangle since it is at right angles to the radius. The entire diagram is rotationally symmetrical, so this distance applies to all points P that are the same distance from the centre of the circle. The right angled triangle (shown opposite) includes an angle of 30° which means the distance from the centre of the circle to point P is r.sin(30) = r/2.

∴ Region where P can be placed so that the chord is longer than the side length of the equilateral triangle is inside a circle of radius r/2, therefore ¼ of the random chords will be longer.

Chord – 3



Another way to produce random chords is to draw a line from the centre of the circle to the circumference. Place a point on the line and have it move along the line to any random location, once again the point represents the midpoint of the chord. The radial arm can be rotated everywhere so that any chord can be generated. This option appears to be remarkable similar to the Chord-2 example.



Navigate to page 1.2.

For this example, progress straight to the simulation. Run the chord-3 program and determine the proportion of chords that are longer than the side length of the equilateral triangle.

Chords in this example are placed randomly along the radial arm from the centre of the circle to the circumference.

A sample of 100 randomly generated chords using this method is shown opposite.

Question: 9.

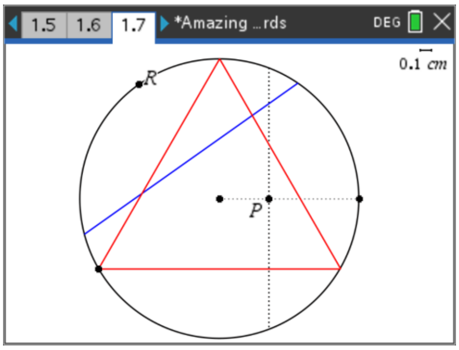
Use the simulation to determine the approximate proportion of chords that are longer than the side length of the inscribed triangle.

Answer: $p \approx 0.5$.

Author: P.Fox

Navigate to page 1.7. Point P is a random point along the radius and represents the midpoint of the chord. This chord can then be rotated using point R.

Move point P along the radius and observe 'ghost' chord (dotted) and the final chord after rotation (blue).



Question: 10.

For what range of points along the radius will point P produce a chord that is longer than the side length of the inscribed triangle?

Answer: $p \approx 0.5$. This can be shown by rotating the equilateral triangle so that one side is vertical (right angles to the line segment). The dotted line is the same length as the rotated chord and aligns with the side length of the triangle when P is at the midpoint of the line segment.

Question: 11.

Assuming that point P is equally likely to land on any point along the radius, what proportion of chords produced this way will be longer than the side length of the inscribed triangle?

Answer: $p \approx 0.5$.

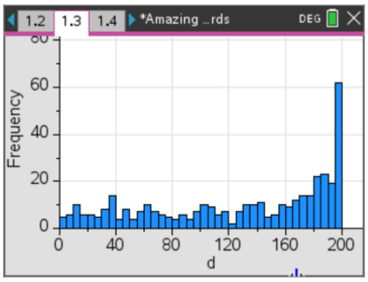
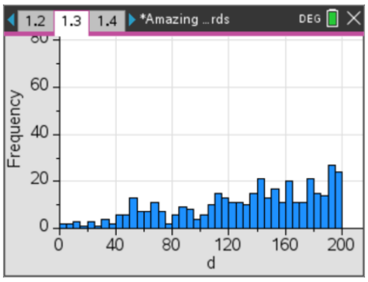
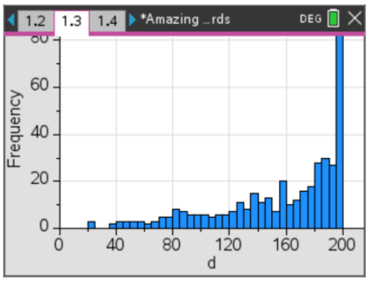
Crazy Chord Discussion

In this activity you have explored three different ways to generate a random chord in a circle and compare the length to the side length of an inscribed triangle. The geometrical 'proofs' produce results consistent with the simulations. Run the different chord#() programs again and explore the distribution of the corresponding lengths created (Page 1.3) and corresponding midpoint distribution (Page 1.4).

Discuss: When a chord is randomly generated in a circle, the probability that it is longer than the side length of an inscribed triangle is …

Answer: All three processes appear reasonable despite producing different, justifiable results! The distribution of the chord lengths is *interesting*. The first method seems the most reasonable way to produce a 'random chord', however it seems to have a tendency to produce particularly long chord lengths and similarly with Method 3. Not surprisingly these two methods also have the higher proportion of chord lengths longer than the side length of the triangle.

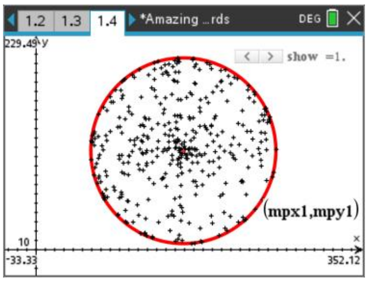
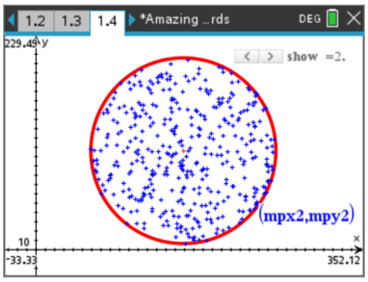
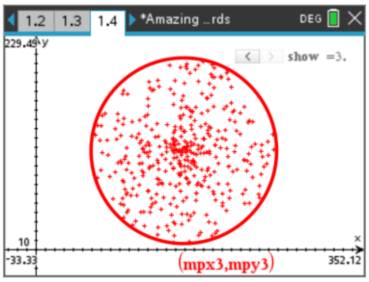
Chord Length Distribution – 400 Trials

Method 1 – Chord1()	Method 2 – Chord2()	Method 3 – Chord3()
		

So, analysis of chord lengths suggests that Method 2 might be the *most reasonable*.

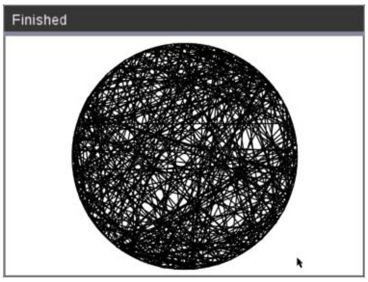
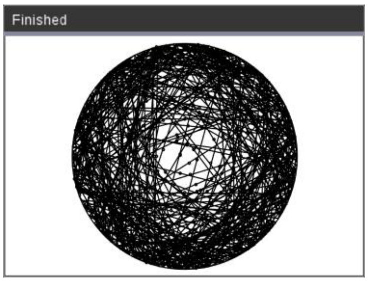
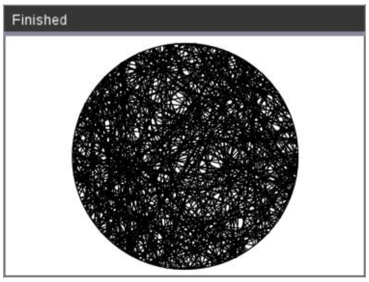
It is also possible to analyse the location of the midpoints.

Midpoint Location – 400 Trials

Method 1 – Chord1()	Method 2 – Chord2()	Method 3 – Chord3()
		

Method 2 appears to have the most uniform distribution of midpoints, not surprising given the method starts by randomly locating the midpoint in the circle.

Chord Image – 400 Trials

Method 1 – Chord1()	Method 2 – Chord2()	Method 3 – Chord3()
		

Method 2 stands out here for the wrong reasons, it appears particularly void of tangents close to the centre of the circle, as evidenced by the very light central region. Method 1 also appears to have some lighter regions, Method 3 certainly seems to have the optimal distribution of chords. Method 2 appears the most favourable when considering the distribution of chord lengths, however it is the least favourable when looking at the distribution of chords in the circle.

Conclusion: The analysis does not identify a preferred method for producing chords. All three results can be accepted and we are left with the conclusion there are three different probabilities, for the 'same' event, a paradox!

Teacher Notes:

Fear not, mathematics is not broken, nor is this activity a comical look at "Proof by contradiction"!
This paradoxical result is known as Betrand's Paradox. All three approaches are legitimate and can be reproduced and confirmed through the combination of simulation and geometric arguments. The paradox (and activity) serve to illustrate that we cannot make assumptions about what is random. It serves as a warning about what we might accept as "computer simulations". We must agree completely on the terms of reference and understand that other 'models' may produce different results. At the conclusion of the activity, it is **HIGHLY** recommended that students watch the two videos produced by Numberphile and Three Blue-One Brown (Grant Sanderson). Hopefully the activity (and videos) will encourage students to watch more videos from both of these extremely popular YouTube channels. Both channels provide numerous, powerful explorations of mathematics and mathematical concepts, often not covered in the curriculum, and by extension, textbooks. Both channels highlight that there is so much more to mathematics.

Video 1: https://www.youtube.com/watch?v=mZBwsm6B280

Video 2: https://www.youtube.com/watch?v=pJyKM-7IgAU