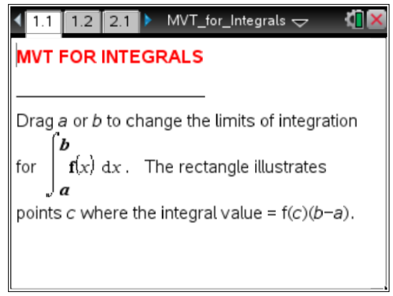
MVT for Integrals

MATH NSPIRED



Math Objectives

- Students will be able to describe the statement of the Mean Value Theorem for Integrals in terms of graphical representation.
- Students will be able to describe how the Mean Value Theorem for Integrals relates to the average value of a function.
- Students will be able to explain why the condition of continuity is needed in the hypothesis of the Mean Value Theorem for Integrals to guarantee the existence of a point at which the function attains its average value.
- Students will construct viable arguments and critique the reasoning of others. (CCSS Mathematical Practice)
- Students will reason abstractly and quantitatively. (CCSS Mathematical Practice)

Vocabulary

- definite integral
- average value of a function

About the Lesson

- This lesson provides a graphical representation of the Mean Value Theorem for Integrals to demonstrate how the average value of a function over an interval is related to the definite integral.
- As a result, students will:
  - Change the endpoints of intervals to make connections between the definite integral (the area bounded by the graph of **f**(x) and the x-axis) and a corresponding rectangular area.
  - Discover that if a function **f**(x) is continuous it is always possible to construct a rectangle of height **f**(c) such that its area is equal to the definite integral and that this value **f**(c) is the average value of the function over the given interval.
  - Use graphical representations of the Mean Value Theorem for Integrals to determine the average value of functions.
  - Recognize conditions under which the existence conclusion of the Mean Value Theorem does not hold.

TI-Nspire™ Navigator™

- Use Class Capture to check students' reasoning about a range of intervals and to explore multiple solutions.
- Use Quick Poll to assess student understanding of the Mean Value Theorem for Integrals.

Activity Materials

- Compatible TI Technologies: TI-Nspire™ CX Handhelds, TI-Nspire™ Apps for iPad®, TI-Nspire™ Software

Tech Tips:

- This activity includes screen captures taken from the TI-Nspire CX handheld. It is also appropriate for use with the TI-Nspire family of products including TI-Nspire software and TI-Nspire App. Slight variations to these directions may be required if using other technologies besides the handheld.
- Watch for additional Tech Tips throughout the activity for the specific technology you are using.
- Access free tutorials at http://education.ti.com/calculators/pd/US/Online-Learning/Tutorials

Lesson Files:

Student Activity

- MVT_for_Integrals_Student.pdf
- MVT_for_Integrals_Student.doc

TI-Nspire document

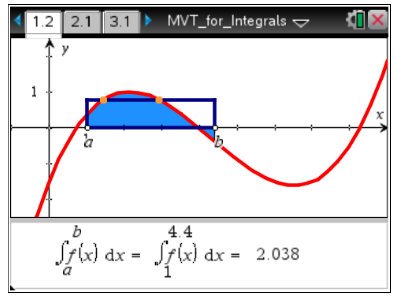
- MVT_for_Integrals.tns

Discussion Points and Possible Answers

Tech Tip: If students experience difficulty dragging a point, check to make sure that they have moved the cursor until it becomes a hand (👆) getting ready to grab the point. Also, be sure that the word *point* appears, not the word *text*. Then press ctrl 👆 to grab the point and close the hand (✊).

The Mean Value Theorem for Integrals states that if **f**(x) is continuous on the interval [a, b], then there exists a number c in the interval (a, b) such that: $\int_a^b \mathbf{f}(x)dx = f(c)(b-a)$.

In this activity, you will explore a visual representation of this theorem and consider some of its implications.



Move to page 1.2.

1. The graph displays the graph of y = **f**(x) on the closed interval [a, b]. Drag the endpoints a and b along the x-axis to change the interval.

   a. Describe the relationship between the shaded region and the definite integral calculation below the graph.

   Answer: The definite integral is the signed area bounded by the graph of y = **f**(x) and the x-axis indicated by the shaded region.

Teacher Tip: This first question is meant as a review or formative assessment of students' understanding of the interpretation of definite integral as representing the signed area between the graph of y = **f**(x) and the x-axis (as one moves left to right). If students have difficulty, you may wish to review this idea or complete the activity Definite Integral before moving on.

   b. Place the endpoint a at 1 and describe the relationships between the shaded region and the displayed rectangle as you move the other endpoint, b, to the right.

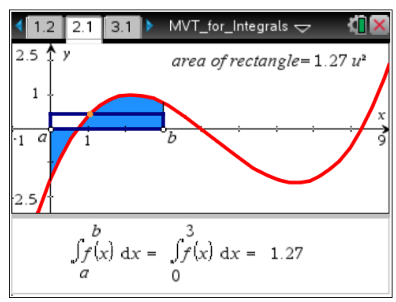
Sample answer: As you move *b* to the right, the area of the shaded region and the rectangle are both increasing until *b* = 4 when the function values become negative. While the rectangle is still above the *x*-axis, the definite integral is now decreasing and the height of the rectangle is decreasing. As you continue moving *b* to the right, the shaded area below the *x*-axis gradually becomes greater than the area above the *x*-axis when *b* is between 6 and 7. From there, the value of the integral becomes negative and the rectangle is below the *x*-axis. As *b* moves farther right, more "negative" area is accumulated, and the rectangle gets larger as the height is increasing in the negative direction.

c. Make a conjecture about the relationship between the rectangle and the definite integral over an interval. Explain how you might justify your conjecture.

Sample answer: The area of the rectangle seems to be equal to the absolute value of the definite integral, with the side of the *x*-axis the rectangle lies on indicating the sign of the definite integral (below is negative, above is positive). Students may suggest calculating the area of the rectangle by using the graph to estimate its height or directly measuring using **Menu > Measurement > Area** and selecting the polygon.

Tech Tip: Select **> Measurement > Area**.

Move to page 2.1.



2. The graph shown is of the same function as the previous page, with the area of the rectangle now displayed in the upper right corner. Drag the endpoints *a* and *b* to evaluate each integral and describe how these values relate to the area of the rectangle in each case:

a. $\int_0^3 \mathbf{f}(x)dx$ b. $\int_1^4 \mathbf{f}(x)dx$ c. $\int_0^6 \mathbf{f}(x)dx$ d. $\int_2^7 \mathbf{f}(x)dx$

Answer: ≈ 1.27 **Answer:** 0 **Answer:** ≈ 2.11 **Answer:** ≈ –1.9

Answer: In each case, the area of the displayed rectangle is the same as the absolute value of the definite integral. It should be noted that for 1a and 1b, the definite integral is positive and the rectangle is above the *x*-axis. In part 1c, when the definite integral is zero, the shaded areas above and below the *x*-axis are equal and no rectangle is formed. Students should note the difference in signs for part 1d. Although the definite integral is negative, the area displayed is positive and the rectangle is below the *x*-axis, which students may interpret as a "negative height."

3. One side of the rectangle is parallel to the *x*-axis and intersects the graph of $y = \mathbf{f}(x)$ at the point given by coordinates $(c, \mathbf{f}(c))$.

   a. Explain how the area of this rectangle is represented in the statement of the Mean Value Theorem for Integrals given above.

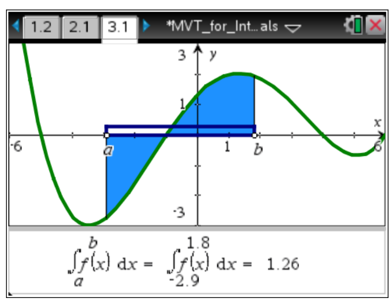
   Answer: The side of the rectangle along the *x*-axis has a length of $b - a$, the same as the length of the interval. The height of the rectangle is the function value $\mathbf{f}(c)$. Thus the expression on the right side of the Mean Value Theorem, $\mathbf{f}(c)(b - a)$ is simply the area of the rectangle. The graph is providing a visual means of representing the Mean Value Theorem for Integrals, that is, it demonstrates that there exists a point *c* in the interval such that the area of the rectangle with height $\mathbf{f}(c)$ and length $b - a$ will be equal to the signed area bounded by $\mathbf{f}(x)$ and the *x*-axis.

   b. The average (or mean) value of a function **f** over a closed interval $[a, b]$ is defined as $\dfrac{\int_a^b \mathbf{f}(x)dx}{b-a}$. The title "Mean Value Theorem for Integrals" suggests that it has something to do with mean value. Explain how the point $(c, \mathbf{f}(c))$ relates to mean value.

   Sample answer: The *x*-value *c* is the point at which $\mathbf{f}(c)$ is equal to the mean value of the function *f* over the closed interval $[a, b]$. If you divide both sides of the equation $\int_a^b \mathbf{f}(x)dx = \mathbf{f}(c)(b-a)$ by $(b - a)$, you get the definition of mean value on the one side of the equation and $\mathbf{f}(c)$ on the other.

   NOTE: Conceptually, the average can be thought of as evenly distributing the function values among all the input values, or spreading out the shaded area evenly over the interval. When this signed area is spread out evenly across the interval, the average value is given by the height of the rectangle, $\mathbf{f}(c)$.

Move to page 3.1.



4. Drag *a* and/or *b* to identify intervals for which the definite integral for this new function $\mathbf{f}(x)$ is positive and for which it is negative.

   a. What do you notice about the rectangle in each case?

   Answer: When the definite integral is positive, the rectangle is above the *x*-axis, and when the definite integral is negative, the rectangle is below the *x*-axis.

b. What does this tell you about the average value of the function over these intervals? Explain.

Answer: When the rectangle is below the *x*-axis, the average value of the function is positive. When the rectangle is above the *x*-axis, the average value of the function is negative.

TI-Nspire Navigator Opportunity: *Class Capture*
See Note 1 at the end of this lesson.

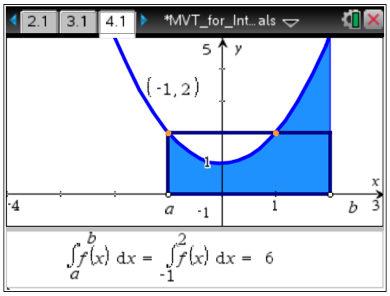
5. Suppose the average value of the function **f**(*x*) is zero.
   a. What would the rectangle look like in this case?

   Answer: When the average value of the function is zero, the rectangle "disappears" or appears as only a thick line on the *x*-axis because the height of the rectangle is zero.

   b. What could you say about the definite integral and the area bounded by **f**(*x*) and the *x*-axis in this case?

   Answer: The definite integral is also zero over this interval.

Move to page 4.1.



6. Explain how you could use the graph to find the average value of this new function over the interval [–1, 2].

   Answer: The average value is 2. This can be determined by moving *a* and *b* to –1 and 2, respectively, and dividing the value of the definite integral, which is 6, by the length of the interval, which is 3. Students may also determine this by noting the *y*-coordinate of the intersection point (–1, 2).

7. Is there an *x*-value in the interval [–1, 2] for which the function value is equal to this average value? Explain how you can determine this.

   Answer: Yes, the function value is 2 at both *x* = –1 and *x* = 1. These values can be found by noting that the side of the rectangle and the graph of **f**(*x*) intersect at the points (–1, 2) and (1, 2).

8. Repeat questions 6 and 7 for the interval [–3, 0].

Answer: The average value is 4. It can be found by dividing the definite integral by the length of the interval $\left(\frac{12}{3} = 4\right)$ or by noting the function value of the intersection point (–1.73, 4). The *x*-coordinate of this intersection point, –1.73, tells where the function value is equal to this average value.

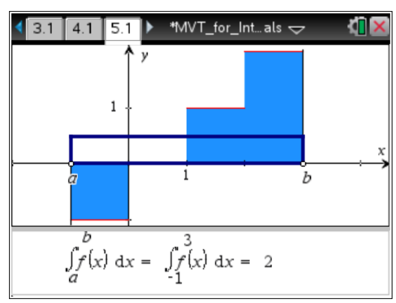
9. Both the Mean Value Theorem and the Mean Value Theorem for Integrals are known as “existence” theorems. Explain what this means in each case.

Answer: The Mean Value Theorem guarantees that there is some place within the interval where the instantaneous rate of change is equal to the average rate of change over that interval. Similarly, the Mean Value Theorem for Integrals guarantees that there is some place in the interval where the function value is equal to the average value of the function over the interval.

Teacher Tip: This question is meant to reinforce key ideas of the Mean Value Theorem for Integrals and draw parallels between the two theorems. If students are having difficulty, you may wish to provide the statement of the Mean Value Theorem given in the answer above and ask students to make a similar statement for the Mean Value Theorem for Integrals. You might also review the MVT through the associated “Mean Value Theorem” activity.

TI-Nspire Navigator Opportunity: *Quick Poll* or *Class Capture*
See Note 2 at the end of this lesson.

Move to page 5.1.



10. Use the graph to find the average value of the greatest integer function on the following intervals and explain how you determined each of these values:
    a. [–1, 4]

    Answer: The average value is 1. Students may take the value of the definite integral, which is 5, and divide by the interval length, which is also 5. They may also note that the intersection point (1.01, 1) appears on the screen.

b. [2, 4]

Answer: The average value is 2.5. Again, students may take the value of the definite integral, 5, and divide by the interval length, 2. They may also simply note that the height of the rectangle appears to be equal to 2.5.

c. [–1, 3]

Answer: The average value is 0.5. The definite integral is 2 divided by 4, which is the length of the interval. In other words, it is the height of the rectangle.

11. Which of these average values are possible function values? Explain.

Answer: Only the first average value of 1 is a possible function value. Because the greatest integer function only produces integer function values, 2.5 and 0.5 are not possible. This is also apparent in the graph. The side of the rectangle does not intersect the function graph when exploring the intervals [2, 4] and [–1, 3].

12. Does this example violate the Mean Value Theorem for Integrals? Explain why or why not.

Answer: No, this is not a violation of the theorem. Because this function is not continuous, it does not meet the initial conditions and thus the Mean Value Theorem for Integrals cannot be applied. It is important to note that while the graphical display still shows it is possible to construct the rectangle whose area is equal to the definite integral, the height of this rectangle is not necessarily a function value. In other words, the existence portion of the Mean Value Theorem for Integrals does not hold. It is not always possible to find a c value in the interval for which $\mathbf{f}(c)(b - a)$ is equal to the definite integral.

Wrap Up

Upon completion of the discussion, the teacher should ensure that students understand:

- How the symbolic representation of the Mean Value Theorem for Integrals can be interpreted graphically as finding the area of a rectangle whose base is the interval and whose height is some value of $\mathbf{f}(x)$ that creates a rectangle with the same area as the signed area bounded by $\mathbf{f}(x)$ and the x-axis. This height, $\mathbf{f}(c)$ in the statement of the theorem, is the average value of the function.
- How to use the Mean Value Theorem for Intervals to determine the average value of a function both graphically and numerically.
- Why the condition that $\mathbf{f}(x)$ be continuous is necessary in order to apply the Mean Value Theorem for Integrals.

Assessment

The following question might be posed through a Quick Poll as an individual or group assessment.

The Mean Value Theorem for Integrals states that if **f**(x) is continuous on the interval [a, b], then there exists a number c in the interval (a, b) such that: $\int_a^b \mathbf{f}(x)dx = \mathbf{f}(c)(b-a)$

a. Solve this algebraic statement for **f**(c).
b. Describe how this expression provides a numeric way of calculating the average value of a function.
c. Describe how this relates to the geometric interpretation of the average function value.

Sample answers:

a. $f(c) = \frac{1}{b-a}\int_a^b f(x)dx$
b. Numerically, the average value of a function can be found by calculating the definite integral over the interval and dividing by the length of the interval.
c. Geometrically, this corresponds to determining the height of a rectangle, whose base is the interval, which would provide an area to the signed area between **f**(x) and the x-axis.

TI-Nspire™ Navigator™

Note 1

Question 4, *Class Capture:* Use *Class Capture* to display several student screens to illustrate a variety of intervals for which the definite integrals are positive. Examples should include intervals for which the function is always positive as well as intervals over which both positive and negative values are obtained to discuss the idea of having a larger area above the x-axis. Similar ideas can be discussed for intervals where the definite integral is negative.

Note 2

Question 9, *Quick Poll (Multiple Choice or Open Response)* or *Class Capture:* You may wish to use question 9 as a *Quick Poll* to assess students' ability to use the graph to determine the average value of the function before moving on.

Additional *Quick Poll* questions:

1. Find another interval for which the average function value is 4.

   Answer: [–3, 3]

2. Find the average value of the function on the interval [–2, 2] when $x \approx \pm 1.15$.

   Answer: ≈ 2.33

   Class Capture may also be used here to answer the above questions or to have students demonstrate a variety of intervals for which the average function value is less than 4 or greater than 4.