Math Objectives

- Students will understand how the Law of Sines is derived.
- Students will understand when the Law of Sines can be used to find missing sides and angles in a triangle.
- Students will be able to solve for missing sides and angles in a triangle using the Law of Sines, when appropriate.
- Students will use appropriate tools strategically (CCSS Mathematical Practice).

Vocabulary

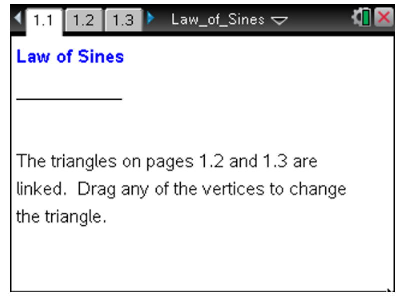
- sine of an angle
- obtuse angle
- acute angle
- right angle
- Ambiguous Case of the Law of Sines

About the Lesson

- This lesson involves exploring the relationship known as the Law of Sines.
- As a result, students will:
  - Manipulate a triangle to observe the equality of ratios of side lengths and sines of opposite angles.
  - Manipulate a triangle to decide when the Law of Sines gives an unambiguous result.
  - Consider the Law of Sines in connection with previous knowledge about right triangle trigonometry.
  - Determine when the Law of Sines can be used to solve for unknown side lengths and angle measures in a triangle, and apply it to solve when possible.

TI-Nspire™ Navigator™ System

- Transfer a File
- Use Screen Capture to examine patterns that emerge.
- Use Quick Poll to assess students' understanding.



TI-Nspire™ Technology Skills:

- Download a TI-Nspire document
- Open a document
- Move between pages
- Grab and drag a point

Tech Tips:

- Make sure the font size on your TI-Nspire handhelds is set to Medium.
- You can hide the function entry line by pressing ctrl G.

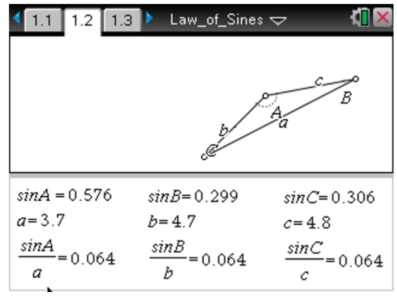
Lesson Files:
Student Activity
Law_of_Sines_Student.pdf
Law_of_Sines_Student.doc

TI-Nspire document
Law_of_Sines.tns

Visit www.mathnspired.com for lesson updates and tech tip videos.

Discussion Points and Possible Answers

Move to page 1.2.



1. This page shows triangle *ABC*, with angles *A*, *B*, and *C*, and corresponding sides opposite those angles whose lengths are *a*, *b*, and *c*, respectively. You can drag any of the vertices to change the triangle.
   a. Drag the vertices to change the shape of the triangle. How does the size of the angle relate to the length of the side opposite the angle? Explain.

   Sample Answers: The larger the angle, the longer the side opposite the angle.

   b. Below the triangle, you will see ratios. What is each ratio comparing?

   Sample Answers: Each ratio compares the value of the sine of one angle to the length of the side opposite that angle.

   c. Drag the vertices, and observe the values of the ratios. What do you observe?

   Sample Answers: For any triangle, the three ratios are equal to each other.

TI-Nspire Navigator Opportunity: *Screen Capture*
See Note 1 at the end of this lesson.

2. Adjust the triangle so that sin *B* = 1.
   a. What is the measure of angle *B*? How do you know? What kind of triangle is triangle ABC?

   Sample Answers: Angle *B* must be 90°. This can be observed through the ,tns document, but it can also be found using the arcsine function, or through familiarity with the unit circle. Triangle ABC is a right triangle.

   b. Use trigonometric ratios to write sin A as a ratio of sides a and b.

   Sample Answers: $\sin A = \frac{a}{b}$.

c. Use the fact that sin *B* = 1 to show that, in this case, $\dfrac{\sin A}{a} = \dfrac{\sin B}{b}$.

Sample Answers: Since sin *B* = 1, $\sin A = \frac{a}{b} = \frac{a}{b} \cdot 1 = \frac{a}{b} \cdot \sin B$.
Thus, $\dfrac{\sin A}{a} = \dfrac{\sin B}{b}$.

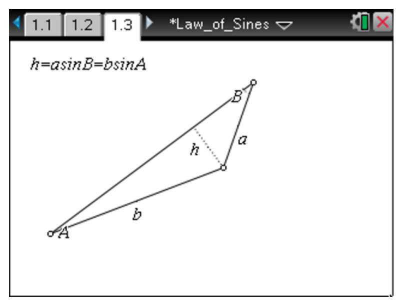
d. Use a similar argument to show that, in this case, $\dfrac{\sin C}{c} = \dfrac{\sin B}{b}$.

Sample Answers: Because *ABC* is a right triangle with hypotenuse *b*, we have $\sin C = \frac{c}{b}$. Since sin *B* = 1, we have $\sin C = \frac{c}{b} = \frac{c}{b} \cdot 1 = \frac{c}{b} \cdot \sin B$, so $\dfrac{\sin C}{c} = \dfrac{\sin B}{b}$.

e. Why must it be true, in this case, that $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$?

Sample Answers: This is true because of the transitive property of equality.

Move to page 1.3.



3. The equality you just showed in question 2 is called the Law of Sines. It is true for all triangles, not just right triangles. Investigate why the Law of Sines is true on Page 1.3.
   a. Move vertex *B* so that angle *C* is obtuse. When *C* is obtuse, the segment *h* appears within the triangle. The segment *h* is called an altitude of the triangle, and it is perpendicular to the side it intersects. Explain why the statement at the top of the screen, $h = a \sin B = b \sin A$ is true.

Sample Answers: The altitude, *h*, is perpendicular to side *c*, so it forms right angles with side *C*. This results in two right triangles, one containing angle *A* and one containing angle *B*, sharing a common side *h*. Then, considering the right triangle containing angle *A*, we have $\sin A = \frac{h}{b}$. This gives $b \sin A = h$. For the right triangle containing angle *B*, we have $\sin B = \frac{h}{a}$, which gives $a \sin B = h$. Therefore, $h = a \sin B = b \sin A$.

   b. Using the fact that $h = a \sin B = b \sin A$, explain why the Law of Sines is true for your triangle.

Sample Answers: Because $a \sin B = b \sin A$, we have $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$.

c. What if *C* is a different obtuse angle? Will this change your results in the previous parts of this problem?

Sample Answers: No. When *C* is obtuse, side length *c* will always be the longest side, so it will be possible to drop a perpendicular side *h*, and the relationship described in part a will always hold.

TI-Nspire Navigator Opportunity: *Screen Capture*
See Note 2 at the end of this lesson.

4. a. Move vertex *B* until angle *C* is a right angle. How can you be certain *C* is a right angle?

Sample Answers: The sine of *C* will be 1. This can be checked by returning to Page 1.2.

b. Does the relationship between *h*, *a*, and *b* still hold when *C* is a right angle? Explain.

Sample Answers: Yes. A perpendicular can still be dropped from vertex *C* to side *c*, creating two right triangles. The reasoning in problem 3 still holds.

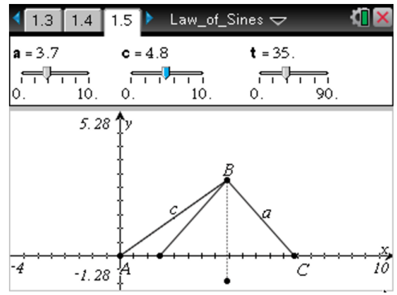
c. What does this tell you about the Law of Sines when *C* is a right angle? Explain.

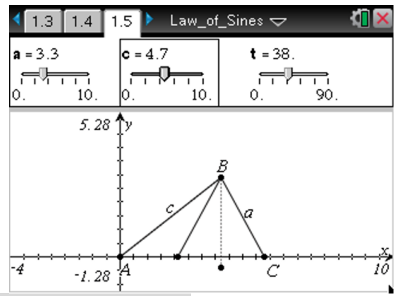
Sample Answers: The law of Sines will still hold when *C* is a right angle.

5. The Law of Sines can be used to find unknown side lengths and angle measures in a triangle, including many triangles that are not right triangles. In order to use the Law of Sines to find unknown side lengths and angle measures in a triangle, what is the minimum information you would need to know about the triangle? Explain.

Sample Answers: You will need to know at least two sides and one of the corresponding angles, or two angles and one of the corresponding sides. This is because the Law of Sines relates two angles and two corresponding sides.

Move to page 1.5.





6. For each part below, set the side lengths and angle on your handheld as specified. How many triangles can you build with each of the specifications? Why is that true?
   a. *c = 3.5, a = 5.8, A = 60°*

   <u>**Sample Answers:**</u> 1.

> **Teacher Tip:** Students might need help interpreting the document. When a single triangle appears, all sides of which are solid lines, this indicates that only one triangle can be constructed with the given specifications. When no complete triangle appears, and the dotted line does not meet the opposite side of the triangle, this indicates that no triangle can be constructed with the given specifications. In the case where two triangles can be constructed, students will see a picture similar to the one below. In this case the two triangles are the small triangle to the left, and the large solid triangle containing the smaller triangle.

   b. *c = 3, a = 7, A = 25°*

   <u>**Sample Answers:**</u> 1.

   c. *c = 5, a = 3.8, A = 40°*

   <u>**Sample Answers:**</u> 2.

   d. *c = 6, a = 3.5, A = 30°*

   <u>**Sample Answers:**</u> 2.

   e. *c = 6, a = 2, A = 30°*

   <u>**Sample Answers:**</u> 0.

TI-Nspire Navigator Opportunity: *Quick Poll*
See Note 3 at the end of this lesson.

7. Fix c = 4 and t = 30°. Change the value of a, and observe the number of triangles that can be made. What is the relationship between side *a*, side *c*, and segment *h* (the perpendicular from vertex *B* to side *b*) when you can build 0, 1, or 2 triangles?

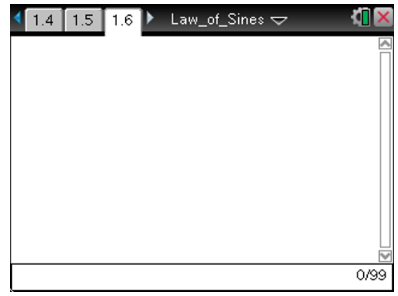
   <u>Sample Answers:</u> When $a < h$, no triangles can be built. The perpendicular, *h*, is the shortest distance from vertex *B* to side *b*. If *a* is shorter than *h*, it will not connect the vertex to the opposite side, and therefore will not create a triangle. When $a \geq c$ and $a \geq h$, exactly 1 triangle can be built. When $h < a < c$, two triangles can be built. Because $a \geq h$, at least one triangle can be built, because this ensures that side *a* will be long enough to connect vertex *B* to side *b*. When $a \geq c$, there is only one direction side *a* can extend: away from side *c*. Otherwise, angle *A* would be changed. Thus in this case, only one triangle can be constructed with the given specifications. However, if $a < c$, it is possible to construct two triangles; one with side *a* extending toward side *c* and one with side *a* extending away from side *c*.

8. What do your results in question 7 tell you about when you can use the Law of Sines to solve for the unknown quantities in a triangle? Forming two triangles is called the Ambiguous Case. Under what circumstances will the Ambiguous Case occur? What quantities should you check before applying the Law of Sines? Explain.

   <u>Sample Answers:</u> The Law of Sines can only be used when exactly one triangle can be built. Prior to applying the Law of Sines, one should find *h* (using the product of the sine of the known angle and the non-corresponding side). One should ensure that *a* is larger than $c \sin A$, but also less than *c*. In general, the product of the non-corresponding side and the sine of the known angle must be smaller than the corresponding side, and the corresponding side must be less than the non-corresponding side.

Teacher Tip: Teachers might want to remind students that other sides and angles of a triangle might be known, and that, while the relationships will hold, the variables might differ. Students might need to practice labeling *h* and defining the relationships to be investigated.

Move to page 1.6.



9. Determine if there are 0, 1, or 2 triangles possible, and use the Law of Sines, if possible, to solve for the missing side lengths and angles in each of the following triangles. If no triangles are possible, explain why not.

   a. *a = 1.2, b = 1. B = 42º*

   <u>Sample Answers:</u> There are two possible triangles. This is an SSA situation, so there are two possibilities for the unknown side lengths and angles:
   A = 53.4°, *C* = 84.6°, *c* = 1.49
   A'=126.6, C' =11.4, c' = .30

   b. *a = 25, B = 39º, C = 112º*

   <u>Sample Answers:</u> This triangle can be solved with Law of Sines, and there is no need to consider the ambiguous case because, by ASA triangle congruence, there can only be one such triangle.
   A = 29°, *b* = 32.45, *c* = 47.81

   c. *a = 2, b = 7, A = 41º*

   <u>Sample Answers:</u> This triangle cannot be solved. $b \sin A \approx 4.6$, but *a* = 2. In this case, no such triangle could be built.

d. *a = 10, c = 12, A = 50º*

   <u>Sample Answers:</u> In this case, $c \sin A \approx 9.2$, so we have *h* < *a*, but *a* < *c*. This means two triangles are possible, so there will not be a unique solution. The possible triangles are:
   C=66.8, B=63.2, b=11.7
   C'=113.2 B'=16.8, b'=3.8

Wrap Up

Upon completion of the lesson, the teacher should ensure that students are able to understand:

- The Law of Sines.
- The Law of Sines can be proved by constructing two right triangles within the original triangle.
- In cases where 0 triangles can be built, the Law of Sines cannot be used.
- In cases where 2 triangles can be built, application of the Law of Sines can be used to find both triangles, but there will not be a unique solution.
- The number of possible triangles can be determined using the product of the sine of the known angle and the non-corresponding side.

Assessment

Teachers can give students more triangles to solve. Teachers might want to include contextual problems to ensure that students experience the applications of the Law of Sines.

TI-Nspire Navigator

Note 1

Name of Feature: Screen Capture

A Screen Capture can be used to show that for all the different triangles investigated by students, the Law of Sines continues to hold.

Note 2

Name of Feature: Screen Capture

A Screen Capture can be used to show multiple student screens and help students to observe that for all obtuse angles C, the same relationship between h and a and b exists.

Note 3

Name of Feature: Quick Poll

A Quick Poll can be used to ensure that students have correctly observed the number of possible triangles given each set of parameters.